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All supersymmetric solutions of minimal supergravity in five dimensions

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Abstract

All purely bosonic supersymmetric solutions of minimal supergravity in five dimensions are classified. The solutions preserve either one half or all of the supersymmetry. Explicit examples of new solutions are given, including a large family of plane-fronted waves and a maximally supersymmetric analogue of the Gödel universe which lifts to a solution of eleven dimensional supergravity that preserves 20 supersymmetries.

Dedicated to the memory of Sonia Stanciu

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1 Introduction

Solutions of higher dimensional supergravity theories have played a key role in elucidating the structure of string theory. Many interesting solutions have been found describing higher dimensional black holes, black branes and their intersections, pp-waves and so on. However, two recent discoveries suggest that higher dimensional gravity may harbour a much richer spectrum of objects that remains to be discovered.

First, it has been suggested that there may exist a family of black brane solutions without translational symmetry [1, 2]. There are no known exact solutions describing such objects but there is numerical evidence [3, 4] that such solutions do exist. Secondly, an exact solution of the five dimensional vacuum Einstein equations has been found that describes an asymptotically flat black hole of topology $S^1 \times S^2$: a rotating black ring [5]. This is the first example of a black hole of non-spherical topology. Furthermore, the existence of this solution implies that the black hole uniqueness theorems cannot be extended to five dimensions, except in the static case [6].

It is tempting to suspect that these new solutions are just the tip of the iceberg, and that many more surprises will be found in higher dimensions. It is clearly desirable to have a better understanding of exact solutions of higher dimensional supergravity theories. Unfortunately, solving the Einstein equations is notoriously difficult even in four dimensions.

Supersymmetric solutions of supergravity theories are of particular importance in string theory because such solutions often have certain stability and non-renormalization properties that are not possessed by non-supersymmetric solutions. For example, it has been possible to give a microscopic description of certain supersymmetric black holes [7]. However, this work relies on the assumption that there is a uniqueness theorem for supersymmetric black holes in four and five dimensions. For supersymmetric rotating black holes (which only seem to exist in five dimensions), this might not be the case if supersymmetric black rings exist. It is therefore of interest to examine the extent to which supersymmetry excludes some of the more exotic solutions of higher dimensional gravity discussed above, or alternatively provides a setting in which they can be studied in more detail. To do so, we would like to know the general nature of supersymmetric solutions of higher dimensional supergravity theories.

Although there are many partial results for $D = 10$ and $D = 11$ supergravities concerning manifolds with special holonomy, various brane solutions etc, a systematic classification of *all* supersymmetric solutions remains a challenging problem. A more modest goal is to attempt a similar classification for simpler supergravity theories, which can be viewed as a truncation of the $D = 10$ or $D = 11$ supergravity theories.

Some time ago, following [8], this was carried out for minimal $N = 2$ supergravity in $D = 4$

by Tod [9]. It was shown that the supersymmetric solutions fall into two classes depending on whether the Killing vector obtained from the Killing spinor is time-like or null. Moreover, the general solution could be obtained explicitly in both cases. In the timelike case, one obtains the Israel-Wilson-Perjes (IWP) class of solutions and the null case consists of pp-waves. There are also supersymmetric solutions with sources, provided the sources saturate a BPS bound relating the energy density to the electric and magnetic charge densities [8], [9]. Some generalizations of this result for other $D = 4$ theories were presented in [10].

The goal of the present paper is to extend this classification to the simplest higher dimensional supergravity theory: the minimal $N = 1$, $D = 5$ supergravity theory constructed in [11]. This is a similar theory in the sense that it has the same number of supercharges and furthermore after dimensional reduction on a circle it gives $N = 2$ supergravity in $D = 4$ coupled to a vector multiplet. However, in five dimensions it is not possible to use the Newman-Penrose formalism adopted in [9, 10] and new techniques are required.

Following [12, 13], the basic strategy is to assume the existence of at least one Killing spinor, and consider the differential forms that can be constructed as bilinear quantities from this spinor. These satisfy a number of algebraic and differential conditions that then can be used to deduce the form of the metric and the gauge fields. It is clear from the outset that for this theory one should not expect to be able to explicitly construct all solutions in closed form, since, for example, a simple class of solutions is the product of $K3$ with a flat time direction with vanishing gauge fields, and the explicit metric on $K3$ is not known. Nevertheless, we are able to give a simple set of rules for the construction of all supersymmetric solutions in this theory.

We find that the supersymmetric solutions fall into two classes, as in [9], depending on whether the Killing vector $\bar{\epsilon}\gamma^{\mu}\epsilon$ obtained from the Killing spinor ϵ is timelike or null. In each class the solutions preserve 1/2 or all of the supersymmetry. In the null case, the general solution can be obtained explicitly. It is a plane-fronted wave, specified by three arbitrary harmonic functions on $\mathbb{R}^3$. This can be contrasted with the situation in the $N = 2$, $D = 4$ theory, where the null solutions are given by pp-waves. In our solution, pp-waves appear merely as a special case specified by two harmonic functions on $\mathbb{R}^3$. Perhaps somewhat surprisingly, our null family of solutions contains some familiar *static* spacetimes such as the supersymmetric magnetic black string solution [14], and its near horizon geometry, $AdS_3 \times S^2$. The point is that these spacetimes are boost invariant and therefore admit a null Killing vector field, and it turns out that this is what is obtained from a Killing spinor.

In the timelike case we find that supersymmetric solutions are specified by the following data: a hyper-Kähler 4-manifold B describing the spatial base geometry orthogonal to the orbits of the Killing vector field; a 1-form connection ω defined locally on B and a function f defined globally on B, satisfying a pair of simple equations. Solutions with non-vanishing ω generically describe rotating, or boosted, spacetimes.

The electrically charged rotating supersymmetric black hole of Beckenridge, Myers, Peet and Vafa (BMPV) [15] can be obtained as a solution of $N = 1$, $D = 5$ supergravity [16]. In our classification, it has base manifold $B = \mathbb{R}^4$. This solution has some rather peculiar properties. For example, although the solution has angular momentum, the angular velocity of the horizon is zero [16]. Furthermore, there are closed timelike curves (CTCs) behind the horizon [15], and if the angular momentum is sufficiently large then the solution no longer describes a black hole but is instead a geodesically complete asymptotically flat, supersymmetric time machine [17]. The appearance of naked CTCs was related to a breakdown in unitarity of the underlying microscopic description in [18].

One result of our investigation is that CTCs seem to be generic for the timelike solutions in five dimensions (this is similar to $D = 4$ as closed time-like curves are generic for the $D = 4$ IWP solutions [19]). It turns out to be rather difficult to find any new solutions which do *not* have closed timelike curves or singularities. One of our solutions is of particular interest owing to its close similarity to Gödel's four dimensional rotating universe solution [20]. Gödel's solution motivated interest in time machines in General Relativity because it is a homogeneous solution with trivial topology $\mathbb{R}^4$ yet contains CTCs through every point. Our five dimensional solution has very similar properties and is slightly simpler than Gödel's.

It is interesting to note that there are some similarities with the equations arising in the time-like case and those in the "Resolution through transgression" series of papers (see [32] for a review). In particular, if the solutions are static then a harmonic function on the hyper-Kähler base appears in the $D = 5$ solution. Generically these are singular solutions in $D = 5$. Stationary solutions, on the other hand, have the harmonic function replaced by a function that solves a Laplace equation modified by the square of a self-dual harmonic two form, and we show that these can lead to non-singular solutions.

For the timelike case with base space given by a Gibbons-Hawking space (the most general with a tri-holomorphic Killing vector), we are able to show that the most general solution is specified by four arbitrary harmonic functions on $\mathbb{R}^3$.

We have examined the further conditions required for our solutions to preserve maximal supersymmetry. For the null case we are led to flat space, $AdS_3 \times S^2$ and a certain plane-wave solution [21]. In the timelike case, flat space, $AdS_2 \times S^3$ and the near horizon geometry of the BMPV solution are all known to be maximally supersymmetric but surprisingly it turns out that the generalized Gödel solution also preserves all eight supercharges.

The maximally supersymmetric timelike solutions just listed all have flat base space. However, maximally supersymmetric solutions can also be obtained from non-flat base spaces. For example, there is a novel construction of $AdS_2 \times S^3$ using a nakedly singular Eguchi-Hanson space and another solution using negative mass Taub-NUT that gives rise to the Gödel spacetime. These examples show that the five dimensional geometry does not uniquely determine the base space of maximally supersymmetric solutions. Furthermore, it turns out that the maximally supersymmetric null solutions have some Killing spinors that correspond to timelike Killing vectors and hence these solutions must lie in *both* classes. In the timelike description, the plane wave arises from a base space describing a smeared distribution of Taub-NUT instantons and $AdS_3 \times S^2$ arises from another singular hyper-Kähler base space. These examples illustrate that physically interesting and regular five dimensional solutions can arise from a pathological base space.

All of the solutions of minimal $D = 5$ supergravity can be uplifted to obtain solutions of $D = 10$ and $D = 11$ supergravity. In general one expects that the uplifted solutions will preserve either 4 or 8 supersymmetries for the generic and the maximally supersymmetric D=5 solutions, respectively. Surprisingly, we show that the Gödel solution uplifts to a solution of $D = 11$ supergravity that preserves 20 supersymmetries. Although general arguments have been put forward for the existence of supergravity solutions preserving all fractions of supersymmetry, and in particular between 1/2 and 1 [22], to date the only such solutions that have been found are in the plane wave category [23]-[28]. This Gödel solution thus constitutes a new class of solution preserving an exotic fraction of supersymmetry.

Our results constitute the first analysis of *all* supersymmetric solutions of a higher dimensional supergravity theory. The results of this work provide encouraging evidence that the strategy of [12, 13] could be used to perform a similar classification of all supersymmetric solutions of other higher dimensional supergravity theories. A key idea of [12, 13] is to relate supersymmetric solutions to different kinds of G-structures and we will discuss this relationship for $D = 5$. One motivation for studying the minimal $D = 5$ supergravity is that it is very similar in structure to 11-dimensional supergravity. Supersymmetric solutions to D=5, N=1 supergravity coupled to matter have been studied in e.g. [29], [30].

The plan of the rest of the paper is as follows. In section 2 we show how various bosonic quantities are constructed from a Killing spinor and derive differential and algebraic relations between these quantities. Section 3 analyses the case when the Killing vector constructed from the Killing spinor is timelike and includes several new solutions. Section 4 carries out a similar analysis when the Killing vector is null. Section 5 discusses maximally supersymmetric solutions. Section 6 discusses the connection with G-structures. Section 7 uplifts the Gödel solution to D=11 supergravity and section 8 concludes. The paper contains two appendices.

2 D=5 supergravity

The bosonic action for minimal supergravity in five dimensions is

$$S = \frac{1}{4\pi G}\int\left(-\frac{1}{4}R*1 - \frac{1}{2}F\wedge *F - \frac{2}{3\sqrt{3}}F\wedge F\wedge A\right), \tag{2.1}$$

We will adopt the conventions of [11] and these are outlined in appendix A[1]. The bosonic equations of motion are

$$\begin{aligned} R_{\alpha\beta} + 2(F_{\alpha\gamma}F_{\beta}{}^{\gamma} - \frac{1}{6}g_{\alpha\beta}F^2) &= 0 \\ d*F + \frac{2}{\sqrt{3}}F\wedge F &= 0 \end{aligned} \tag{2.2}$$

where $F^2 \equiv F_{\alpha\beta}F^{\alpha\beta}$. A bosonic solution to the equations of motion is supersymmetric if it admits a a super-covariantly constant spinor obeying

$$\left[D_\alpha + \frac{1}{4\sqrt{3}}\left(\gamma_\alpha{}^{\beta\gamma} - 4\delta_\alpha^\beta\gamma^\gamma\right)F_{\beta\gamma}\right]\epsilon^a = 0. \tag{2.3}$$

where ϵ^a is a symplectic Majorana spinor. We shall call such spinors Killing spinors. Our strategy for determining the most general bosonic supersymmetric solutions[2] is to analyse the differential forms that can be constructed from commuting Killing spinors. We first investigate algebraic properties of these forms, and then their differential properties.

From a single commuting spinor ϵ^a we can construct a scalar f, a 1-form V and three 2-forms $\Phi^{ab} \equiv \Phi^{(ab)}$:

$$f\epsilon^{ab} = \bar{\epsilon}^a\epsilon^b, \tag{2.4}$$

$$V_\alpha\epsilon^{ab} = \bar{\epsilon}^a\gamma_\alpha\epsilon^b, \tag{2.5}$$

$$\Phi^{ab}_{\alpha\beta} = \bar{\epsilon}^a\gamma_{\alpha\beta}\epsilon^b, \tag{2.6}$$

f and V are real, but Φ^{11} and Φ^{22} are complex conjugate and Φ^{12} is imaginary. It is useful to work with three real two-forms defined by

$$\Phi^{(11)} = X^{(1)} + iX^{(2)}, \qquad \Phi^{(22)} = X^{(1)} - iX^{(2)}, \qquad \Phi^{(12)} = -iX^{(3)}. \tag{2.7}$$

These quantities give a total of $1 + 5 + 3\times 10 = 36$ real degrees of freedom. To understand this, note that ϵ^a has a total of 8 real components. The product $\epsilon^a_\alpha\epsilon^b_\beta$ is symmetric in (a,α)

[1]The sign of the Chern-Simons term corrects that appearing in [11].

[2]Note that there are spacetimes admitting a Killing spinor that do not satisfy the equations of motion. These can be viewed as solutions of the field equations with additional sources, and supersymmetry imposes conditions on these sources. For example, in the case of solutions with a timelike Killing vector, the source must be a charged dust with charge density equal to the mass density, analogous to the sources in [8, 9]. Here we will restrict ourselves to solutions of the field equations without sources.

and (b,β). A symmetric 8×8 matrix has 36 components, corresponding to the 36 degrees of freedom obtained above (by projecting with C, $C\gamma_\alpha$ and $C\gamma_{\alpha\beta}$) . Having said this, $\epsilon^a_\alpha\epsilon^b_\beta$ is *not* a general symmetric 8×8 matrix because it has rank 1 and only 8 real degrees of freedom. It follows that there should be algebraic relations between the above quantities that reduce the number of independent components from 36 to 8.

It will be useful to record some of these identities which can be obtained from various Fierz identities. We first note that

$$V_\alpha V^\alpha = f^2 \tag{2.8}$$

which implies that V is timelike, null or zero. The final possibility can be eliminated by noting $2V_0 = {\epsilon_a}^\dagger\epsilon_a > 0$ in any region in which the Killing spinor is non-vanishing. We will work in such a region, and analytically continue the metric to the fixed point sets of the isometry generated by V at which V vanishes. In later sections we will analyse the time-like and null cases separately. We also have

$$X^{(i)}\wedge X^{(j)} = -2\delta_{ij} f * V, \tag{2.9}$$

$$i_V X^{(i)} = 0, \tag{2.10}$$

$$i_V * X^{(i)} = -f X^{(i)}, \tag{2.11}$$

$$X^{(i)}_{\gamma\alpha} X^{(j)\gamma}{}_\beta = \delta_{ij}\left(f^2\eta_{\alpha\beta} - V_\alpha V_\beta\right) + \epsilon_{ijk} f X^{(k)}_{\alpha\beta}\ , \tag{2.12}$$

where $\epsilon_{123}=+1$ and, for a vector Y and p-form A, $(i_Y A)_{\alpha_1\ldots\alpha_{p-1}} \equiv Y^\beta A_{\beta\alpha_1\ldots\alpha_{p-1}}$. Finally, it is useful to record

$$V_\alpha\gamma^\alpha\epsilon^a = f\epsilon^a\ , \tag{2.13}$$

and

$$\Phi^{ab}_{\alpha\beta}\gamma^{\alpha\beta}\epsilon^c = 8f\epsilon^{c(a}\epsilon^{b)}. \tag{2.14}$$

For the remainder of the paper we will usually suppress the symplectic indices on the spinors.

We now turn to the differential conditions that can be obtained by assuming that ϵ is a Killing spinor. We differentiate f, V, Φ in turn and use (2.3). Starting with f we find

$$df = -\frac{2}{\sqrt{3}} i_V F. \tag{2.15}$$

Taking the exterior derivative and using the Bianchi identity for F then gives

$$\mathcal{L}_V F = 0, \tag{2.16}$$

where $\mathcal{L}$ denotes the Lie derivative. Next, differentiating V gives

$$D_\alpha V_\beta = \frac{2}{\sqrt{3}} F_{\alpha\beta} f + \frac{1}{2\sqrt{3}}\epsilon_{\alpha\beta\gamma\delta\epsilon} F^{\gamma\delta} V^\epsilon, \tag{2.17}$$

which implies $D_{(\alpha}V_{\beta)} = 0$ and hence V is a Killing vector [31]. Combining this with (2.16) implies that V is the generator of a symmetry of the full solution (g, F). Rewriting (2.17) as

$$dV = \frac{4}{\sqrt{3}} f F + \frac{2}{\sqrt{3}} * (F \wedge V) \tag{2.18}$$

and then taking the exterior derivative gives

$$0 = \frac{2}{\sqrt{3}} \left[\mathcal{L}_V * F - i_V \left(d * F + \frac{2}{\sqrt{3}} F \wedge F \right) \right]. \tag{2.19}$$

The first term on the right hand side vanishes as a consequence of equations (2.16) and the fact that V is a Killing vector. The second term vanishes[3] if one imposes the equation of motion for F.

Finally, differentiating $X^{(i)}$ gives

$$D_\alpha X^{(i)}_{\beta\gamma} = -\frac{1}{\sqrt{3}} \left[2F_\alpha{}^\delta \left(*X^{(i)} \right)_{\delta\beta\gamma} - 2F_{[\beta}{}^\delta \left(*X^{(i)} \right)_{\gamma]\alpha\delta} + \eta_{\alpha[\beta} F^{\delta\epsilon} \left(*X^{(i)} \right)_{\gamma]\delta\epsilon} \right], \tag{2.20}$$

which implies

$$dX^{(i)} = 0, \tag{2.21}$$

and

$$d * X^{(i)} = -\frac{2}{\sqrt{3}} F \wedge X^{(i)}. \tag{2.22}$$

To make further progress we will examine separately the case in which the Killing vector is time-like and the case in which it is null in the two following sections. More precisely, the case in which f vanishes everywhere will be analyzed in section 4. If f does not vanish everywhere then pick a point p at which $f \neq 0$. By continuity, f must be non-zero in a neighbourhood $\mathcal{U}$ of p. The analysis of section 3 will give the general solution in $\mathcal{U}$ and this can then be analytically extended to the whole spacetime.

3 The timelike case

3.1 Introduction

In this section we shall consider the case in which f is non-zero and hence V is a timelike Killing vector field. Equation (2.12) implies that the 2-forms $X^{(i)}$ are all non-vanishing. Introduce coordinates such that $V = \partial/\partial t$. The metric can then be written locally as

$$ds^2 = f^2(dt + \omega)^2 - f^{-1} h_{mn} dx^m dx^n \tag{3.1}$$

[3]Note that this calculation indicates the consistency between the sign of the Chern-Simons term in the supergravity action and the sign and factors appearing in the Killing spinor equation.

where we have assumed, essentially with no loss of generality, $f > 0$ (we shall return to this point shortly). The metric $f^{-1}h_{mn}$ is obtained by projecting the full metric perpendicular to the orbits of V. The manifold so defined will be referred to as the base space B and we will deduce that h_{mn} is a hyper-Kähler metric.

Define

$$e^0 = f(dt + \omega) \tag{3.2}$$

and if η defines a positive orientation on B then we use $e^0 \wedge \eta$ to define a positive orientation for the D=5 metric. The two form $d\omega$ only has components tangent to the base space and can therefore be split into self-dual and anti-self-dual parts with respect to the metric h_{mn}:

$$f d\omega = G^+ + G^- \tag{3.3}$$

where the factor of f is included for convenience. Equations (2.15) and (2.18) can now be solved for F, giving

$$F = \sqrt{3}\left(-\frac{1}{2}f^{-2}V \wedge df + \frac{1}{6}G^+ + \frac{1}{2}G^-\right), \tag{3.4}$$

which can also be written

$$F = \frac{\sqrt{3}}{2}de^0 - \frac{1}{\sqrt{3}}G^+. \tag{3.5}$$

Note that for all previously known solutions, including the BMPV black hole [15, 16] (which will be briefly reviewed below), $d\omega$ is anti-self-dual so the second term on the right-hand-side of (3.5) is absent. The Bianchi identity and equation of motion for F imply the following equations:

$$dG^+ = 0 \tag{3.6}$$

and

$$\Delta f^{-1} = \frac{4}{9}(G^+)^2 \tag{3.7}$$

where Δ is the Laplacian in the metric h, and $(G^+)^2 \equiv (1/2)(G^+)_{mn}(G^+)^{mn}$ where the indices here are raised with h_{mn}.

Equation (2.10) implies that the 2-forms $X^{(i)}$ can be regarded as 2-forms on the base space and Equation (2.11) implies that they are anti-self-dual:

$$*_4 X^{(i)} = -X^{(i)}, \tag{3.8}$$

where $*_4$ denotes the Hodge dual associated with the metric h_{mn}. Equation (2.12) can be written

$$X^{(i)}{}_m{}^p X^{(j)}{}_p{}^n = -\delta^{ij}\delta_m{}^n + \epsilon_{ijk}X^{(k)}{}_m{}^n \tag{3.9}$$

where indices $m, n, \ldots$ have been raised with h^{mn}, the inverse of h_{mn}. This equation shows that the $X^{(i)}$'s satisfy the algebra of imaginary unit quaternions. Furthermore, we find that (2.20) yields

$$\nabla_m X^{(i)}_{np} = 0, \tag{3.10}$$

where ∇ is the Levi-Civita connection associated with h_{mn}. Combined with (3.9) this shows that the base space does indeed admit an integrable hyper-Kähler structure.

We have exhausted the content of the equations satisfied by the bosonic quantities. We next examine the Killing spinor equation itself and find that it imposes no further conditions. In an orthonormal basis with time direction e^0 given by (3.2), equation (2.13) implies

$$\gamma^0 \epsilon = \epsilon. \tag{3.11}$$

Using this with (A.3) one can show that $\gamma^{ij}\epsilon$ is anti-self-dual with respect to the base space metric and hence

$$G^+_{ij}\gamma^{ij}\epsilon = 0. \tag{3.12}$$

Here $i, j = 1 \ldots 4$ refer to components in a basis orthonormal with respect to h_{mn} (but γ^{ij} is still defined in terms of the five dimensional gamma matrices). Using these results, the 0 component of the Killing spinor equation then implies that ϵ is time-independent. The spatial components are then solved if

$$\epsilon(t, x) = f^{1/2}\eta(x) \tag{3.13}$$

where $\eta(x)$ is a covariantly constant spinor on the hyper-Kähler base space. Now any hyper-Kähler space admits covariantly constant chiral spinors satisfying $\gamma^{1234}\eta = \eta$ if the Kähler-forms are anti-self dual. Noting that this chirality condition is actually a consequence of (3.11) implies that (3.11) is the only projection imposed on the Killing spinors and we deduce that the configurations preserve at least 1/2 of the supersymmetry.

We have imposed the Bianchi identity for F and the F field equation, so we should also check whether the Einstein equations are satisfied. They are in fact automatically satisfied as one can deduce from the integrability condition for the Killing spinor equation presented in appendix B. From there we have that

$$\begin{aligned} E_{\mu\nu}V^\nu &= 0 \\ E_{\mu\nu}E_\mu{}^\nu &= 0 \qquad \text{no} \quad \text{sum} \quad \text{on} \quad \mu \end{aligned} \tag{3.14}$$

where $E_{\mu\nu} = 0$ is equivalent to the Einstein equations. Working in the orthonormal frame, the first condition implies $E_{00} = E_{i0} = 0$ and the second then implies $E_{ij} = 0$.

In summary, the above analysis shows that the general supersymmetric solution in the stationary case with $f > 0$ is determined by a hyper-Kähler base 4-manifold B with metric

h_{mn} and an orientation chosen so that the hyper-Kähler two-forms are anti-self-dual, together with a globally defined function f and locally defined 1-form connection ω on B. Writing $fd\omega = G^- + G^+$, we have $dG^+ = 0$ and also $\Delta f^{-1} = (4/9)(G^+)^2$. The field strength is then determined as in (3.4). These solutions[4] preserve at least 1/2 of the supersymmetry, with Killing spinors satisfying (3.11).

When $f < 0$ an identical analysis reveals that the most general supersymmetric solution is simply obtained from this solution by simply taking $t \to -t, \omega \to -\omega$ and reversing the orientation on the base manifold B. In other words

$$\begin{aligned} ds^2 &= |f|^2(dt+\omega)^2 - |f|^{-1}h_{mn}dx^m dx^n \\ F &= -\left(\frac{\sqrt{3}}{2}de^0 - \frac{1}{\sqrt{3}}G^-\right). \end{aligned} \tag{3.17}$$

with $e^0 = |f|(dt+\omega)$ and positive orientation $e^0 \wedge \eta$, where η is a positive orientation on the base manifold, is a supersymmetric solution with Killing spinors satisfying $\gamma^0\epsilon = -\epsilon$ provided that h is a hyper-Kähler metric with self-dual hyper-Kähler two-forms, $|f|$ is a globally defined function and writing $|f|d\omega = G^+ + G^-$ one demands $dG^- = 0$ and also $\Delta|f|^{-1} = (4/9)(G^-)^2$.

In subsequent subsections we will construct explicit solutions working with the case $f > 0$ for definiteness. The subsections can be essentially read independently of each other. Subsection 3.2 discusses static solutions; 3.3 solutions when the base space is compact; 3.4 the maximally supersymmetric Gödel-type solution; 3.5 some further solutions with a flat hyper-Kähler base space including a quick review of the BMPV black hole; 3.6 focuses on solutions with base space Eguchi Hanson and Taub-NUT as there are some similarities with the "Resolution Through Transgression" papers; 3.7 the general solution for the case that the base space has a Gibbons-Hawking metric [33] and a discussion of how the IWP solutions of $N = 2$, $D = 4$ supergravity can be obtained via dimensional reduction.

3.2 Static solutions

Proposition. The stationary Killing vector field V is hyper-surface orthogonal if, and only, if

[4] Note that given a hyper-Kähler metric, the equations to be solved can be obtained from varying an action functional on B. To see this we first introduce the field strength $\mathcal{F} = d\omega$ and then note that f must satisfy

$$d(f\mathcal{F}^+) = 0, \qquad \Delta f^{-1} = \frac{4}{9}f^2(\mathcal{F}^+)^2 \tag{3.15}$$

where $\mathcal{F}^+ = \frac{1}{2}(\mathcal{F} + *_4\mathcal{F})$. These equations follow from varying the action on B

$$S = \int_B d^4x\sqrt{h}\left(\frac{1}{2}(\partial\sigma)^2 - \frac{4}{9}\sigma^{-1}(\mathcal{F}^+)^2\right) \tag{3.16}$$

with respect to ω and σ, where $\sigma = f^{-1}$.

$G^- = 0$.

Proof. It is easy to show that V is hyper-surface orthogonal if, and only if, $d\omega = 0$. Clearly $d\omega = 0$ implies $G^- = 0$. Conversely, assume $G^- = 0$. Closure of G^+ then gives

$$df \wedge d\omega = 0 \quad \Rightarrow \quad df \wedge G^+ = 0. \tag{3.18}$$

The dual of this gives

$$\partial_m f G^{+mn} = 0, \tag{3.19}$$

so that

$$(df \wedge G^+)_{mnp} G^{+np} = 0 \quad \Rightarrow \quad \partial_m f \left(G^+\right)^2 = 0. \tag{3.20}$$

Hence if $G^- = 0$ then either $df = 0$ or $G^+ = 0$. In the former case, the equation (3.7) for f implies $G^+ = 0$. Hence $G^- = 0$ implies $G^+ = 0$ and therefore $d\omega = 0$.

It follows from this proposition that if $G^- = 0$ then, at least locally, there will be a function $\lambda(x)$ such that $\omega = d\lambda$. A coordinate transformation $t = t' - \lambda(x)$ then brings the metric to a manifestly static form. The solution can then be written

$$\begin{aligned} ds^2 &= f^2 dt^2 - f^{-1} h_{mn} dx^m dx^n, \\ F &= \frac{\sqrt{3}}{2} df \wedge dt, \end{aligned} \tag{3.21}$$

where f^{-1} is a harmonic function on the base space. For a harmonic function with a single pointlike source on a flat base space this gives the non-rotating black hole solution.

Note that these static solutions have vanishing magnetic charge. This does not mean that there are no supersymmetric magneto-static solutions but simply that the static Killing vector of such a solution cannot be written as $\bar{\epsilon}\gamma_\alpha\epsilon$ with ϵ a Killing spinor. For example, it will be shown in section 4 that the magnetic black string solution corresponds to the null case $f \equiv 0$.

3.3 Solutions with compact base space

If the base space is compact then it must be K3 (with self-dual curvature) or T^4. Suppose f is smooth and non-vanishing on the base space. Integrating equation (3.7) yields $G^+ = 0$ and hence f^{-1} is harmonic. However there are no non-trivial smooth harmonic functions on a compact manifold so f must be constant. By rescaling t, ω and h_{mn} we can set $f = 1$. This leaves

$$d\omega = G^-, \tag{3.22}$$

so G^- is closed and anti-self-dual and therefore harmonic. Taking the wedge product of this equation with G^- and integrating over the base space shows that if G^- is non-zero then ω cannot be globally defined.

The only anti-self-dual harmonic forms on K3 (with self-dual curvature) or T^4 are the complex structures so it follows that

$$d\omega = 4\gamma J, \tag{3.23}$$

where J is an anti-self-dual complex structure and γ a constant. The field strength is

$$F = 2\sqrt{3}\gamma J, \tag{3.24}$$

and the metric is

$$ds^2 = (dt + \omega)^2 - h_{mn} dx^m dx^n. \tag{3.25}$$

These solutions describe rotating closed universes containing a constant magnetic field. The case of T^4 can be discussed more explicitly. Local coordinates can be chosen such that

$$h_{mn} dx^m dx^n = (dx^1)^2 + (dx^2)^2 + (dx^3)^2 + (dx^4)^2, \tag{3.26}$$

and

$$J = \left(dx^1 \wedge dx^2 - dx^3 \wedge dx^4\right). \tag{3.27}$$

One could then take

$$\omega = 2\gamma \left(x^1 dx^2 - x^2 dx^1 - x^3 dx^4 + x^4 dx^3\right). \tag{3.28}$$

This is clearly not globally defined on T^4. Rather than discuss this further we shall discuss the analogous solution on the covering space of T^4, namely $\mathbb{R}^4$.

3.4 A supersymmetric analogue of the Gödel universe

The supersymmetric Gödel universe has metric given by (3.25),(3.26),(3.28) and gauge field given by (3.24). For simplicity we let $\gamma = 1/4$. On the base space $\mathbb{R}^4$, write $x^1 = r_1 \cos\phi_1$, $x^2 = r_1 \sin\phi_1$, $x^3 = r_2 \cos\phi_2$, $x^4 = r_2 \sin\phi_2$ and using ω as in (3.28) the solution can be written[5]

$$\begin{aligned} ds^2 &= \left(dt + \frac{1}{2}(r_1^2 d\phi_1 - r_2^2 d\phi_2)\right)^2 - \left(dr_1^2 + r_1^2 d\phi_1^2 + dr_2^2 + r_2^2 d\phi_2^2\right) \\ F &= \frac{\sqrt{3}}{2}(r_1 dr_1 \wedge d\phi_1 - r_2 dr_2 \wedge d\phi_2). \end{aligned} \tag{3.29}$$

The metric has coordinate singularities at $r_1 = 0$ and $r_2 = 0$ but these can be removed by going back to Cartesian coordinates. Note that $\partial/\partial\phi_i$ is timelike for $r_i > 2$ so this solution has closed timelike curves in those regions. Note that the signature remains Lorentzian. This is very similar to what happens in the four dimensional Gödel universe [20]. There are further similarities between our solution and Gödel's. The Gödel solution is defined on a manifold of topology

[5]This solution has previously appeared in [34] as a footnote.

$\mathbb{R}^4$. Ours has topology R^5. The matter content of Gödel's solution consists of pressureless dust balanced by a negative cosmological constant. Calculating the energy-momentum tensor for the $F_{\mu\nu}$ of our solution one finds that it (and all the solutions of the previous section) has vanishing pressure and constant energy density proportional to γ^2, i.e., the electromagnetic field has the same energy-momentum as pressureless dust. In addition, just like the Gödel universe, this solution is homogeneous; it was shown in [35] that the near-horizon geometry of the BMPV black hole is also homogeneous. It is straightforward to verify that the following vectors in cartesian co-ordinates are Killing vectors:

$$
\begin{aligned}
&V = \frac{\partial}{\partial t}, \\
B_1 &= \frac{\partial}{\partial x^1} - \frac{x^2}{2}\frac{\partial}{\partial t}, \qquad & B_2 &= \frac{\partial}{\partial x^2} + \frac{x^1}{2}\frac{\partial}{\partial t}, \\
B_3 &= \frac{\partial}{\partial x^3} + \frac{x^4}{2}\frac{\partial}{\partial t}, \qquad & B_4 &= \frac{\partial}{\partial x^4} - \frac{x^3}{2}\frac{\partial}{\partial t}, \\
R_1 &= x^1\frac{\partial}{\partial x^2} - x^2\frac{\partial}{\partial x^1}, \qquad & R_2 &= x^3\frac{\partial}{\partial x^4} - x^4\frac{\partial}{\partial x^3}
\end{aligned}
\tag{3.30}
$$

which act transitively. The last two Killing-vectors generate a $U(1)\times U(1)$ group of rotations in $\mathbb{R}^4$. In fact, as we will see later (see equation (3.43)) this is actually enlarged to an a $SU(2)\times U(1)$ group of rotations, giving a 9 parameter family of isometries, the same number as for $AdS_2\times S^3$ or $AdS_3\times S^2$.

Surprisingly, this solution is maximally supersymmetric preserving all 8 supersymmetries of the theory. Explicitly, using the obvious frame $(e^0, e^i) = (dt+\omega, dx^i)$, the Killing spinors are given by

$$
\epsilon = \theta^+ + (1 + J_{ij}x^i\gamma^j)\theta^- \tag{3.31}
$$

where $\theta^\pm$ are constant spinors satisfying $\gamma^0\theta^\pm = \pm\theta^\pm$.

We can determine the symmetry superalgebra using the method of [36] (see also [37, 38, 39]). This is a Lie-superalgebra whose even subspace $\mathcal{B}$ is spanned by the above Killing-vectors and the odd subspace $\mathcal{F}$ by the Killing spinors. The bilinear map $\mathcal{B}\times\mathcal{B}\to\mathcal{B}$ is the Lie-bracket of the above vector fields. The map $\mathcal{B}\times\mathcal{F}\to\mathcal{F}$ is obtained from the Lie-derivative of the Killing spinors with respect to the Killing vectors and the map $\mathcal{F}\times\mathcal{F}\to\mathcal{B}$ is deduced from the Killing-vectors obtained by squaring the Killing spinors. It is straightforward to write all of these maps out explicitly, but we shall just record the last map. We first note that for any two Killing spinors ϵ, ϵ' we can construct a Killing vector via $K^\mu = \bar{\epsilon}\gamma^\mu\epsilon'$. If we let $\epsilon = \theta^+ + (1+J_{ij}x^i\gamma^j)\theta^-$ and $\epsilon' = \rho^+ + (1+J_{ij}x^i\gamma^j)\rho^-$ we find

$$
\bar{\epsilon}\gamma^\alpha\epsilon' E_\alpha = \overline{\theta^+}\rho^+ V + \overline{\theta^-}\rho^-(-V - 2R_1 + 2R_2) + (\overline{\theta^+}\gamma^i\rho^- + \overline{\theta^-}\gamma^i\rho^+)B_i \tag{3.32}
$$

where E_α are the vector fields dual to the frame introduced above.

3.5 Further solutions with flat base-space

Let us now present some additional new solutions with base space $\mathbb{R}^4$ that admit at least an $SU(2)$ sub-group of isometries of the $SO(4)$ rotation group of $\mathbb{R}^4$. It will be useful to work with $SU(2)$ Euler-angles, which we introduce via

$$\begin{aligned} x^1 + ix^2 &= r\cos\frac{\theta}{2}e^{i(\frac{\psi+\phi}{2})} \\ x^3 + ix^4 &= r\sin\frac{\theta}{2}e^{i(\frac{\psi-\phi}{2})} \end{aligned} \tag{3.33}$$

with $0 \leq \theta < \pi$, $0 \leq \phi < 2\pi$ and $0 \leq \psi < 4\pi$. We also work with left-invariant one-forms σ^i_R satisfying $d\sigma^i_R = 1/2\epsilon^{ijk}\sigma^j_R \wedge \sigma^k_R$ and right-invariant one-forms σ^i_L satisfying $d\sigma^i_L = -1/2\epsilon^{ijk}\sigma^j_L \wedge \sigma^k_L$. The subscripts refer to the fact that, for example, the left invariant one-forms σ^i_R are dual to right vector fields that generate right actions. Explicit expressions can be found in appendix A. A positive orientation is fixed by $dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4 = dr \wedge (\frac{r}{2})\sigma^1_R \wedge (\frac{r}{2})\sigma^2_R \wedge (\frac{r}{2})\sigma^3_R = dr \wedge (\frac{r}{2})\sigma^1_L \wedge (\frac{r}{2})\sigma^2_L \wedge (\frac{r}{2})\sigma^3_L$. The flat metric on $\mathbb{R}^4$ is given by

$$ds^2 = dr^2 + \frac{r^2}{4}[(\sigma^1)^2 + (\sigma^2)^2 + (\sigma^3)^2] \tag{3.34}$$

for either left or right invariant one-forms.

Let us begin by recording the rotating BMPV solution [15, 16]. We write the flat base using left-invariant one-forms σ^i_R. As previously noted it has $G^+ = 0$ and is given by:

$$\begin{aligned} f^{-1} &= 1 + \frac{\mu}{r^2} \\ \omega &= \frac{j}{2r^2}\sigma^3_R \end{aligned} \tag{3.35}$$

The ADM mass and angular momentum are given by

$$\begin{aligned} M &= \frac{3\pi\mu}{4G} \\ J &= -\frac{j\pi}{2G} \end{aligned} \tag{3.36}$$

This angular momentum corresponds to equal rotation in the 1-2 and 3-4 planes with opposite sign. Recall that all closed timelike curves are hidden behind the horizon at $r = 0$ providing that $|j| \leq \mu^{3/2}$.

A simple generalization of the BMPV black hole solution with $G^+ \neq 0$ can be obtained by considering the general ansatz

$$\omega = \Psi(r)\sigma^3_R \tag{3.37}$$

and assuming $f = f(r)$. Demanding that G^+ is closed implies that $f(r\Psi' + 2\Psi) = \chi r^2$ for constant χ. Solving (3.7) we then find

$$\begin{aligned} f^{-1} &= \lambda + \frac{\mu}{r^2} + \frac{\chi^2}{9} r^2 \\ \Psi &= \frac{j}{2r^2} + \frac{\chi\mu}{2} + \frac{\chi\lambda}{4} r^2 + \frac{\chi^3}{54} r^4 \end{aligned} \tag{3.38}$$

for constant λ, μ, j. This solution has

$$G^+ = \frac{\chi}{4} d(r^2 \sigma^3_R) = \chi(dx^1 \wedge dx^2 + dx^3 \wedge dx^4) \tag{3.39}$$

Supposing $\lambda \neq 0$ we can always choose $\lambda = 1$ by rescaling the radial and time co-ordinates. We note that the solution will have closed time-like curves when $\Psi^2 f^2 - f^{-1} r^2/4$ is positive. When $\chi = 0$ we return to the rotating black hole solution. When $\chi \neq 0$, the solution is no longer asymptotically flat and is a rotating universe with both electric and magnetic fields and with closed time-like curves.

Further new solutions with flat base space can be found by working with right invariant 1-forms on $SU(2)$, σ^i_L. We now look for solutions with

$$\omega = \Psi(r) \sigma^3_L \tag{3.40}$$

and $f = f(r)$. Following the same steps as above we find the solution

$$\begin{aligned} f^{-1} &= \lambda + \frac{\mu}{r^2} + \frac{\chi^2}{27 r^6} \\ \Psi(r) &= \gamma r^2 - \chi\left(\frac{\lambda}{4r^2} + \frac{\mu}{6r^4} + \frac{\chi^2}{270 r^8}\right) \end{aligned} \tag{3.41}$$

where λ, μ, χ are again constant. For this solution G^+ is given by

$$G^+ = -\frac{\chi}{4} d\left[\frac{1}{r^2} \sigma^3_L\right] \tag{3.42}$$

Let us now discuss this solution in more detail. We first restrict to $\lambda = 1$ which can be achieved by rescaling the radial and time co-ordinates when $\lambda \neq 0$. It appears that these solutions are non-singular at $r = 0$. For example, if one calculates F^2, which from Einstein's equations is proportional to the Ricci scalar, one finds for $\chi \neq 0$ it goes to zero like r^4, while for $\chi = 0$ it goes to a constant. Other curvature invariants also appear to be regular at $r = 0$.

Note also that if we set $\mu = \chi = 0$ and $\gamma \neq 0$, then the solution is given by

$$\begin{aligned} ds^2 &= (dt + \gamma r^2 \sigma^3_L)^2 - (dr^2 + \frac{r^2}{4}[(\sigma^1_L)^2 + (\sigma^2_L)^2 + (\sigma^3_L)^2]) \\ F &= \frac{\sqrt{3}\gamma}{2} d(r^2 \sigma^3_L) \end{aligned} \tag{3.43}$$

which is just the generalised Gödel solution introduced above. We already have shown that the solution has a 7 parameter family of isometries including $U(1)\times U(1)$ rotations in $\mathbb{R}^4$. In the co-ordinates of this section, it is clear that the $U(1)\times U(1)$ symmetry is enlarged to $SU(2)\times U(1)$ corresponding to right $SU(2)$ actions and a left $U(1)$ action (in the 3 direction).

Alternatively, we may set $\gamma = 0$, $\mu \neq 0$, $\chi \neq 0$. The metric is then asymptotically flat. The ADM mass and angular momentum are given by

$$\begin{aligned} M &= \frac{3\pi\mu}{4G} \\ J &= \frac{\chi\pi}{4G} \end{aligned} \tag{3.44}$$

Here the angular momentum corresponds to equal rotation in the 1-2 and 3-4 plane since $d\omega$ is self-dual[6]. Furthermore, it is clear from examining the sign of $f^2\Psi^2 - \frac{1}{4}r^2 f^{-1}$ that there are closed timelike curves for all values of $\mu > 0$ within $0 < r < r_{\text{crit}}$ provided that $\chi \neq 0$. By analysing the behaviour of massive test particles it appears that this geometry is geodesically complete and that $r = 0$ exhibits a repulson behaviour. Moreover by tuning the angular momentum of the test particle it can approach $r = 0$ arbitrarily closely and thus enter the time-machine. The global causal structure of this spacetime is similar to that of the over-rotating BMPV solution presented in [17]. However, unlike the general family of rotating BMPV solutions, tuning M and J in the region $M > 0$, $J > 0$ does not alter the causal structure of the spacetime.

There are a number of straightforward generalisations of the new solutions we have presented here. For example, we could replace (3.37),(3.40) by $\Psi_i(r)\sigma^i$.

3.6 Rotating Eguchi-Hanson and Taub-NUT

When $G^+ = 0$ the function f^{-1} is harmonic on the hyper-Kähler base. When $G^+ \neq 0$ it is modified by the square of a closed, self-dual and hence harmonic form via equation (3.7). This is reminiscent of the equations arising in the "Resolution Through Transgression" papers (for a review see [32]). In particular an interesting regular generalisation of both the Eguchi Hanson space and Taub-NUT space with flux was constructed in section 3 of [40]. We therefore examine the Eguchi-Hanson and the Taub-NUT cases in more detail, finding regular supersymmetric solutions, albeit with closed time-like curves.

Let us first consider the Eguchi-Hanson case. The metric is given by

$$ds^2 = W^{-1}dr^2 + \frac{r^2}{4}((\sigma_L^1)^2 + (\sigma_L^2)^2) + W\frac{r^2}{4}(\sigma_L^3)^2 \tag{3.45}$$

where

$$W = 1 - \frac{a^4}{r^4} \tag{3.46}$$

[6]One can get a solution with the same quantum numbers as the black hole solution (3.35) by taking $t \to -t, \phi \leftrightarrow \psi$. This corresponds to switching to a solution with $f < 0$, as described at the end of section 3.1.

This is a regular space provided that the range of ψ is $0 \le \psi \le 2\pi$ and of r is $a \le r \le \infty$ and $r = a$ is the S^2 bolt. We choose positive orientation to be given by $dr \wedge \sigma_L^1 \wedge \sigma_L^2 \wedge \sigma_L^3$ so that the three Kähler forms are anti-self-dual. If $\omega = 0$ then the harmonic function f^{-1} is constant or singular. The singular solutions can be resolved in the following sense. We choose

$$G^+ = -\frac{\chi}{4} d(r^{-2}\sigma_L^3) \tag{3.47}$$

and hence need to solve

$$\Delta f^{-1} = \frac{8\chi^2}{9r^8} \tag{3.48}$$

where Δ is the Laplacian with respect to the Eguchi-Hanson metric, and one should note a key sign difference with the similar equation in [40]. The general solution is given by

$$f^{-1} = \lambda - \frac{\chi^2}{9a^4r^2} + \delta \log \frac{(r^2 - a^2)}{(r^2 + a^2)} \tag{3.49}$$

where λ, δ are arbitrary integration constants. If we seek solutions of the form

$$\omega = \Psi(r)\sigma_L^3 \tag{3.50}$$

we find

$$\Psi = -\frac{\chi\lambda}{4r^2} + \frac{\chi^3}{54r^4a^4} + \frac{\delta\chi}{4r^2a^4}[(r^4 - a^4)\log\frac{(r^2 - a^2)}{(r^2 + a^2)} + 2a^2r^2] + \gamma r^2 \tag{3.51}$$

A regular solution can be obtained by first setting setting $\delta = 0$ to get

$$\begin{aligned} f^{-1} &= \lambda - \frac{\chi^2}{9a^4r^2} \\ \Psi &= -\frac{\chi\lambda}{4r^2} + \frac{\chi^3}{54r^4a^4} + \gamma r^2 \end{aligned} \tag{3.52}$$

Restricting to $\chi^2 \le \lambda 9a^6$, f^{-1} will be non-zero for $a \le r \le \infty$. By calculating F^2 it appears that this five-dimensional solution is regular, provided that we restrict χ as stated. To eliminate closed time-like curves as $r \to \infty$ we set $\gamma = 0$. However, it seems that there are always closed time-like curves near $r = a$.

It is interesting to note that we can find analogous solutions if we start with a singular hyper-Kähler space given by (3.45) with $W = 1 + b^4/r^4$. In this case there is no reason to take ϕ or ψ to have particular ranges. If we again choose (3.47),(3.50) the solution is given by

$$\begin{aligned} f^{-1} &= \lambda + \frac{\chi^2}{9b^4r^2} + \delta \arctan\frac{r^2}{b^2} \\ \Psi &= -\frac{\chi\lambda}{4r^2} - \frac{\chi^3}{54r^4b^4} - \frac{\delta\chi}{4r^2b^4}[(r^4 + b^4)\arctan\frac{r^2}{b^2} + b^2r^2] + \gamma r^2 \end{aligned} \tag{3.53}$$

These solutions appear to be singular in general. Note that if we take $0 \le \phi \le 2\pi$ and $0 \le \psi \le 4\pi$ then the Eguchi-Hanson space is asymptotically Euclidean and the five dimensional solution is asymptotically flat if the constants of integration are chosen appropriately.

Surprisingly, if we set $\delta = \lambda = \gamma = 0$ we obtain the maximally supersymmetric $AdS_2 \times S^3$ solution as we will show in section 5.

Let us now consider the base space to be Taub-NUT space. The metric is given by

$$ds^2 = \frac{(r+a)}{(r-a)}dr^2 + (r^2 - a^2)((\sigma_R^1)^2 + (\sigma_R^2)^2) + 4a^2\frac{(r-a)}{(r+a)}(\sigma_R^3)^2 \tag{3.54}$$

When a is positive, the range of ψ is $0 \le \psi \le 4\pi$, and that of r is $a \le r \le \infty$, so that the topology of the space is $\mathbb{R}^4$. If we take positive orientation to be given by $adr \wedge \sigma_R^1 \wedge \sigma_R^2 \wedge \sigma_R^3$ then the hyper-Kähler forms are anti-self-dual. We now choose

$$G^+ = \chi d\left[\frac{(r-a)}{(r+a)}\sigma_R^3\right] \tag{3.55}$$

and hence

$$\Delta f^{-1} = \frac{8\chi^2}{9(r+a)^4} \tag{3.56}$$

where Δ is the Laplacian with respect to the Taub-NUT metric. The solution to this is given by

$$f^{-1} = \lambda - \frac{2\chi^2}{9a(r+a)} + \frac{\delta}{r-a} \tag{3.57}$$

As above we let

$$\omega = \Psi(r)\sigma_R^3 \tag{3.58}$$

and find

$$\Psi(r) = -\frac{16\chi^3 a}{27(r+a)^2(r-a)} + \frac{2\chi(2\chi^2 - 9\delta a + 18\lambda a^2)}{9(r+a)(r-a)} - \frac{4\chi\lambda a}{(r-a)} + \gamma\frac{(r+a)}{(r-a)} \tag{3.59}$$

In order to construct a regular solution, we set $\delta = 0$ and choose $\gamma = \chi\lambda - \chi^3/27a^2$ to get

$$\begin{aligned} f^{-1} &= \lambda - \frac{2\chi^2}{9a(r+a)} \\ \Psi &= \frac{\chi(r-a)(27\lambda a^2(r+a) - \chi^2(r+5a))}{27a^2(r+a)^2} \end{aligned} \tag{3.60}$$

Then f^{-1} will be non-zero for $a \le r \le \infty$ and note that $\Psi(a) = 0$, indicating the regularity of the solution. It is straightforward to check that F^2 is also regular everywhere, providing that we restrict χ as mentioned. Again this solution has closed time-like curves.

Once again we can build solutions from the singular negative mass Taub-NUT space by letting a be negative. For example, if we take (3.60) with $\lambda = 0$ we will show in section 5 that we get a maximally supersymmetric solution that is in fact the Gödel solution.

3.7 Solutions with Gibbons-Hawking base space

In this subsection, the equations for f and ω will be examined in more detail for the case of a Gibbons-Hawking base space. The solutions of the last three subsections will comprise special cases. It has been shown [41] that if a four dimensional hyper-Kähler manifold admits a triholomorphic Killing vector field, that is, a Killing vector field L that preserves the complex structures ($\mathcal{L}_L X^{(i)} = 0$), then it must be a Gibbons-Hawking [33] metric:

$$\begin{aligned} ds^2 &= H^{-1}\left(dx^5 + \chi_i dx^i\right)^2 + H dx^i dx^i, \\ \nabla \times \boldsymbol{\chi} &= \nabla H. \end{aligned} \tag{3.61}$$

The Killing vector field is $\partial/\partial x^5$. ∇ is the flat connection on the Euclidean 3-space with coordinates x^i and H is harmonic on this space. The complex structures are given by [41]

$$X^{(i)} = \left(dx^5 + \chi_j dx^j\right) \wedge dx^i - \frac{1}{2} H \epsilon_{ijk} dx^j \wedge dx^k. \tag{3.62}$$

Anti-self-duality of these forms fixes the orientation of the base space so that the volume form is

$$H dx^5 \wedge dx^1 \wedge dx^2 \wedge dx^3. \tag{3.63}$$

Examples of Gibbons-Hawking metrics are: flat space ($H = 1$ or $H = 1/|\mathbf{x}|$), Taub-NUT space ($H = 1 + 2M/|\mathbf{x}|$) and the Eguchi-Hanson space ($H = 2M/|\mathbf{x}| + 2M/|\mathbf{x} - \mathbf{x_0}|$) [41].

If the Killing vector $\partial/\partial x^5$ is a Killing vector of the full five dimensional spacetime (i.e. if f and ω are independent of x^5) then the equations for f and ω can be solved explicitly. Write

$$\omega = \omega_5 \left(dx^5 + \chi_i dx^i\right) + \omega_i dx^i, \tag{3.64}$$

and introduce an orthonormal basis

$$e^5 = H^{-1/2}\left(dx^5 + \chi_i dx^i\right), \qquad e^i = H^{1/2} dx^i. \tag{3.65}$$

Then

$$G^{\pm} = -\frac{1}{2} A_i^{\pm} e^5 \wedge e^i \mp \frac{1}{4} \epsilon_{ijk} A_k^{\pm} e^i \wedge e^j, \tag{3.66}$$

where

$$\mathbf{A}^{\pm} = H^{-1} f \left[H \nabla \omega_5 \mp \omega_5 \nabla H \mp \nabla \times \boldsymbol{\omega}\right]. \tag{3.67}$$

In equations such as this written in three-dimensional vector notation, $\boldsymbol{\omega}$ refers to the three-vector with components ω_i. Closure of G^+ reduces to

$$\nabla \times \mathbf{A}^+ = 0, \tag{3.68}$$

and

$$\nabla \cdot (H\mathbf{A}^+ + \boldsymbol{\chi} \times \mathbf{A}^+) = 0. \tag{3.69}$$

The first of these yields

$$\mathbf{A}^+ = \nabla \rho, \tag{3.70}$$

for some locally defined function ρ. Substituting into the second gives

$$\nabla^2 (H\rho) = 0, \tag{3.71}$$

and hence

$$\rho = 3KH^{-1} \tag{3.72}$$

for some harmonic function K. The equation for f reduces to

$$\nabla^2 f^{-1} = \frac{2}{9} H \left(\nabla \rho\right)^2 = \nabla^2 \left(K^2 H^{-1}\right), \tag{3.73}$$

and hence

$$f^{-1} = K^2 H^{-1} + L, \tag{3.74}$$

where L is another harmonic function. It remains to solve for ω_5 and ω_i. Substituting the above results into (3.67) gives

$$H \nabla \omega_5 - \omega_5 \nabla H - \nabla \times \boldsymbol{\omega} = 3\left(K^2 + LH\right) \nabla \left(KH^{-1}\right). \tag{3.75}$$

Taking the divergence of this gives the integrability condition

$$\nabla^2 \omega_5 = 3H^{-1} \nabla \cdot \left[\left(K^2 + LH\right) \nabla \left(KH^{-1}\right)\right] = \nabla^2 \left(H^{-2} K^3 + \frac{3}{2} H^{-1} KL\right), \tag{3.76}$$

with solution

$$\omega_5 = H^{-2} K^3 + \frac{3}{2} H^{-1} KL + M, \tag{3.77}$$

where M is an arbitrary harmonic function. Substituting the solution back into (3.75) then gives an equation that determines $\boldsymbol{\omega}$ up to a gradient (which can be absorbed into t). The above analysis yields the general solution for which the base space admits a tri-holomorphic Killing vector field that extends to a Killing vector field of the five dimensional spacetime. It is specified by four arbitrary harmonic functions H, K, L, and M. Solutions with $G^+ \neq 0$ are much more complicated than those with $G^+ = 0$ (i.e. $K \propto H$), which are specified by three harmonic functions H, f^{-1} and ω_5.

It is worth remarking that the same solution can be derived under rather weaker assumptions, namely that there exists a spacelike Killing vector field of the five dimensional spacetime that commutes with V and leaves the three complex structures invariant.

The solutions with a flat base space that were discussed above can be easily recovered in this framework. Introduce spherical polar coordinates (R, θ, ϕ) on the three dimensional flat part of the metric. Choosing

$$H = \frac{1}{R}, \qquad \chi = \cos\theta d\phi, \tag{3.78}$$

gives a flat base space. Let $R = r^2/4$ and $x^5 = \psi$ and the metric takes the form

$$ds^2 = dr^2 + \frac{r^2}{4}\left(d\psi^2 + d\phi^2 + 2\cos\theta d\psi d\phi + d\theta^2\right). \tag{3.79}$$

The coordinates (θ, ϕ, ψ) are Euler angles on S^3. The solutions constructed using the left-invariant forms on $SU(2)$ have $\omega = \Psi(r)\left(d\psi + cos\theta d\phi\right)$ and hence $\omega_5 = \Psi$, $\boldsymbol{\omega} = 0$. The full solution is obtained by taking all of the harmonic functions to be spherically symmetric. The solutions constructed using the right-invariant forms have $\omega = \Psi(r)\left(d\phi + \cos\theta d\psi\right)$ and hence $\omega_5 = \Psi(r)\cos\theta$ and $\omega_i dx^i = \Psi(r)\sin^2\theta d\phi$. An example is the Godel solution, which has $G^+ = 0$ and is therefore specified by three harmonic functions: $f^{-1} = 1$, $H = 1/R$, $\omega_5 \propto R\cos\theta$.

The form of the Gibbons-Hawking metric lends itself naturally to dimensional reduction, so the above solution yields a large class of solutions of the theory obtained by KK reduction of the minimal five dimensional supergravity theory, namely $N = 2$, $D = 4$ supergravity coupled to a vector multiplet. It is interesting to see how the solutions of pure $N = 2$, $D = 4$ can be embedded in the five dimensional theory (for *maximally* supersymmetric solutions, this was done in [42]). To this end, consider a general five dimensional metric admitting a spacelike Killing vector field $\partial/\partial x^5$ and write the metric as

$$ds^2 = e^{\alpha\phi} ds_R^2 - e^{\beta\phi}\left(dx^5 + \mathcal{A}\right)^2, \tag{3.80}$$

where ds_R^2 is the line element of the four dimensional Lorentzian metric and $\mathcal{A}$ is a one-form potential on the reduced space. The constants α and β are chosen such that the reduced metric is in the Einstein frame, with a canonically normalized scalar ϕ. Write the five dimensional vector potential as

$$A = A' + \theta dx^5. \tag{3.81}$$

where A' is another one-form potential on the reduced space. The reduced theory has two scalars ϕ and θ and two one-forms A' and $\mathcal{A}$. We want to truncate to get the pure $N = 2$, $D = 4$ theory, so we have to set these scalars to zero. Consistency requires that their equations of motion are satisfied, which gives

$$\frac{3}{4} *_4 G \wedge G + *_4 F' \wedge F' = 0, \tag{3.82}$$

and

$$\frac{\sqrt{3}}{2} *_4 F' \wedge G - F' \wedge F' = 0, \tag{3.83}$$

where $G = d\mathcal{A}$ and $F' = dA'$. The orientation η of the reduced spacetime has been chosen such that $dx^5 \wedge \eta$ is *negatively* oriented in five dimensions because this is what happens for our Gibbons-Hawking solutions. These equations are both satisfied by choosing

$$G = -\frac{2}{\sqrt{3}} *_4 F', \tag{3.84}$$

which also eliminates the vector $\mathcal{A}$ as an independent field (its equation of motion is satisfied using the Bianchi identity for F'). Finally we are left with a theory whose equations of motion can be derived from the action

$$S_4 = \frac{1}{4\pi G_4} \int \left(-\frac{1}{4} R_4 *_4 1 - \frac{2}{3} *_4 F' \wedge F' \right), \tag{3.85}$$

which is indeed the action for the bosonic sector of pure $N = 2$, $D = 4$ supergravity. To summarize, a five dimensional solution with metric of the form (3.80) can be reduced to give a solution of $N = 2$, $D = 4$ supergravity provided $\phi = 0$, $\theta = 0$ (equivalently, $F_{\mu 5} = 0$) and equation (3.84) is satisfied.

Let's apply this to our solutions with Gibbons-Hawking base space. The metric can be written

$$ds^2 = \Lambda^{-1} f H^{-1} \left(dt + \omega_i dx^i \right)^2 - f^{-1} H dx^i dx^i - \Lambda \left(dx^5 + \chi_i dx^i - \frac{f^2 \omega_5}{\Lambda} \left(dt + \omega_i dx^i \right) \right)^2, \tag{3.86}$$

where

$$\Lambda = f^{-1} H^{-1} - f^2 \omega_5^2. \tag{3.87}$$

It can be verified that all of the consistency conditions for the dimensional reduction are satisfied if we choose the harmonic functions L and M such that

$$f^{-1} = \frac{K^2}{H} + H, \qquad \omega_5 = \frac{K}{H} f^{-1}, \tag{3.88}$$

and

$$\nabla \times \boldsymbol{\omega} = 2K \nabla H - 2H \nabla K. \tag{3.89}$$

The reduced metric can be written

$$ds_4^2 = |V|^2 \left(dt + \omega_i dx^i \right) - |V|^{-2} dx^i dx^i, \tag{3.90}$$

where

$$V^{-1} = H + iK, \tag{3.91}$$

$\boldsymbol{\omega}$ is given by

$$\nabla \times \boldsymbol{\omega} = i \left(\bar{V}^{-1} \nabla V^{-1} - V^{-1} \nabla \bar{V}^{-1} \right), \tag{3.92}$$

and the field strength is

$$F' = F = -\frac{\sqrt{3}}{4}\nabla_i \left(V + \bar{V}\right) \tilde{e}^0 \wedge \tilde{e}^i - \frac{\sqrt{3}}{8} i\epsilon_{ijk}\nabla_k \left(V - \bar{V}\right) \tilde{e}^i \wedge \tilde{e}^j, \tag{3.93}$$

where $\tilde{e}^0 = |V|\left(dt + \omega_i dx^i\right)$ and $\tilde{e}^i = |V|^{-1}dx^i$ is an orthonormal basis for the four dimensional metric. This is precisely the form of the IWP metric given in [9]. So the entire timelike class of supersymmetric solutions of the $N = 2$, $D = 4$ theory can be obtained by reduction of a subset of our solutions with Gibbons-Hawking base space. Note that the four dimensional metric is static if $\boldsymbol{\omega} = 0$, which requires $K \propto H$, which is true if, and only if, $G^+ = 0$. For example, setting $K = 0$ gives $\boldsymbol{\omega} = \omega_5 = 0$, $f^{-1} = H$ and the five dimensional metric is

$$ds^2 = H^{-2}dt^2 - H^2 dx^i dx^i - \left(dx^5 + \chi_i dx^i\right)^2, \tag{3.94}$$

which give the electrostatic Majumdar-Papapetrou solutions in four dimensions. Taking $H = 1/|\mathbf{x}|$ gives a flat base space and the four dimensional metric is $AdS_2 \times S^2$ (the five dimensional metric is $AdS_2 \times S^3$). $H = 1 + 1/|\mathbf{x}|$ gives a Taub-NUT base space and the four dimensional metric is extremal Reissner-Nordstrom. Multi-centre Taub-NUT gives multi-centre Reissner-Nordstrom. Eguchi-Hanson gives a two centre $AdS_2 \times S^2$ solution, and multi-centre Eguchi-Hanson gives multi-centre $AdS_2 \times S^2$. Taking K to be a non-vanishing multiple of H just corresponds to a duality rotation of the four dimensional gauge field.

4 The null case

4.1 The general solution

In this section we shall find all solutions of minimal $N = 1$, $D = 5$ supergravity for which the function f introduced in section 2 vanishes everywhere.

First introduce coordinates as follows. From (2.18) it can be seen that V satisfies $V \wedge dV = 0$ and is therefore hypersurface-orthogonal. Hence there exist functions u and H such that

$$V = H^{-1}du. \tag{4.1}$$

A second consequence of (2.17) is

$$V \cdot DV = 0, \tag{4.2}$$

so V is tangent to affinely parametrized geodesics in the surfaces of constant u. One can choose coordinates (u, v, y^m), $m = 1, 2, 3$, such that v is the affine parameter along these geodesics, and hence

$$V = \frac{\partial}{\partial v}. \tag{4.3}$$

The metric must take the form:

$$ds^2 = H^{-1}\left(\mathcal{F}du^2 + 2dudv\right) - H^2\gamma_{mn}\left(dy^m + a^m du\right)\left(dy^n + a^n du\right), \tag{4.4}$$

where the quantities H, $\mathcal{F}$, γ_{mn} and a^m depend on u and y^m only (because V is Killing). Note that there is a lot of gauge freedom remaining. For example, a coordinate transformation of the form $y \to y'(u, y)$ could be used to eliminate a^m. However, this freedom will be more useful shortly.

Equations (2.10) and (2.11) imply that $X^{(i)}$ can be written

$$X^{(i)} = X^{(i)}_m du \wedge dy^m. \tag{4.5}$$

Closure of $X^{(i)}$ then gives

$$\partial_{[m} X^{(i)}_{n]} = 0, \tag{4.6}$$

and hence locally there exist functions $x^i(u, y)$ such that

$$X^{(i)}_m = \partial_m x^i. \tag{4.7}$$

Now we can exploit the freedom to do a coordinate transformation $y \to y'(u, y)$ by choosing $x^i(u, y)$ as our new coordinates. The metric takes the same form as above but with x^i replacing y^m. In these coordinates,

$$X^{(i)} = du \wedge dx^i. \tag{4.8}$$

Equation (2.12) now gives

$$\gamma_{ij} = \delta_{ij}, \tag{4.9}$$

so in these coordinates, the surfaces of constant u and v are flat. The full metric can be written

$$ds^2 = H^{-1}\left(\mathcal{F}du^2 + 2dudv\right) - H^2(d\mathbf{x} + \mathbf{a}du)^2, \tag{4.10}$$

where bold letters denote three dimensional quantities, e.g., $(\mathbf{a})_i \equiv a^i$, with no distinction between "up" and "down" indices for such objects.

It is convenient to introduce a null basis of 1-forms as follows

$$e^+ = V = H^{-1}du, \qquad e^- = dv + \frac{1}{2}\mathcal{F}du, \qquad e^i = H\left(dx^i + a^i du\right). \tag{4.11}$$

These obey the orthogonality relations

$$e^\alpha \cdot e^\beta = \eta^{\alpha\beta}, \tag{4.12}$$

where $\eta^{+-} = \eta^{-+} = 1$, $\eta^{ij} = -\delta_{ij}$ and other components vanish. We also choose $\epsilon_{+-123} = 1$.

Equation (2.15) implies

$$F = F_{+i} e^+ \wedge e^i + \frac{1}{2} F_{ij} e^i \wedge e^j. \tag{4.13}$$

Substituting into equation (2.18) gives

$$F_{ij} = -\frac{\sqrt{3}}{2} H^{-2} \epsilon_{ijk} \nabla_k H, \tag{4.14}$$

where $\nabla_k \equiv \partial/\partial x^k$. Equation (2.20) gives

$$F_{+i} = -\frac{1}{2\sqrt{3}} H \epsilon_{ijk} \nabla_j a_k. \tag{4.15}$$

Hence the field strength is

$$F = -\frac{H^{-2}}{2\sqrt{3}} \epsilon_{ijk} \nabla_j \left(H^3 a_k \right) du \wedge dx^i - \frac{\sqrt{3}}{4} \epsilon_{ijk} \nabla_k H dx^i \wedge dx^j. \tag{4.16}$$

The Bianchi identity reduces to

$$\nabla^2 H = 0, \tag{4.17}$$

$$\partial_u \nabla H = \frac{1}{3} \nabla \times \left(H^{-2} \nabla \times \left(H^3 \mathbf{a} \right) \right), \tag{4.18}$$

The equation of motion for the field strength turns out to be identically satisfied.

Equation (2.13) implies that the Killing spinor satisfies

$$\gamma^+ \epsilon = 0. \tag{4.19}$$

Writing out the Killing spinor equation using the above expressions for the connection and field strength, and using (4.19), one finds that it reduces to

$$\partial_\mu \epsilon = 0. \tag{4.20}$$

Hence the Killing spinor is constant. Since the only restriction is equation (4.19), it follows that, in the null case, as in the timelike case, all supersymmetric solutions preserve at least half of the supersymmetry.

The above analysis yields the general spacetime that admits a constant Killing spinor and satisfies the equations for the field strength. However, the function $\mathcal{F}(u, x)$ is still completely unrestricted so it is necessary to look at the Einstein equations for further information. As in the time-like case we can deduce a lot from the integrability conditions discussed in appendix B. Working in the above basis, (B.5) implies that $E_{\alpha-} = 0$ and (B.6) give $E_{+i} = E_{ij} = 0$. Hence we just need to impose the $++$ component of the Einstein equation, which gives

$$\nabla^2 \mathcal{F} = 2H^2 D_u W_{ii} + 2H W_{(ij)} W_{(ij)} + \frac{2}{3} H W_{[ij]} W_{[ij]}, \tag{4.21}$$

where

$$D_u \equiv \partial_u - a_i \nabla_i, \tag{4.22}$$

and

$$W_{ij} = D_u H \delta_{ij} - H \nabla_j a_i. \tag{4.23}$$

The solution is now specified as follows. First pick a harmonic function $H(u, \mathbf{x})$. Next consider equation (4.18). The general solution will be the sum of a particular integral and the general solution of the homogeneous equation

$$\nabla \times \left(H^{-2} \nabla \times \left(H^3 \mathbf{a} \right) \right) = 0. \tag{4.24}$$

This equation can be integrated to give

$$\nabla \times \left(H^3 \mathbf{a} \right) = H^2 \nabla \phi, \tag{4.25}$$

for some function $\phi(u, \mathbf{x})$. The integrability condition for this is

$$0 = \nabla \cdot \left(H^2 \nabla \phi \right) = H \nabla^2 (H \phi), \tag{4.26}$$

and hence

$$\phi = K H^{-1} \tag{4.27}$$

for some harmonic function $K(u, \mathbf{x})$. Next, equation (4.25) can be integrated to determine $H^3 \mathbf{a}$ up to a gradient. This gradient arises from the gauge freedom $v \to v + g(u, \mathbf{x})$ and therefore can be removed (see the next subsection). So the general solution to equation (4.18) involves a single additional harmonic function K. Finally, equation (4.21) can be solved to determine $\mathcal{F}$ up to another arbitrary harmonic function $\mathcal{F}_0$. Hence the general solution involves three arbitrary u-dependent harmonic functions H, K and $\mathcal{F}_0$.

Recall that a spacetime is said to be a *plane-fronted wave* if it can be foliated by a family of hypersurfaces $u =$ constant such that du is null, geodesic and free of expansion, rotation and shear. This is indeed the case for our solution. In other words the general null supersymmetric solution is always a plane-fronted wave (even though it can be static in special cases, such as the magnetic string). A plane-fronted wave is said to be a *plane-fronted wave with parallel rays* (pp-wave) if du is also covariantly constant. For our solution, this occurs if, and only if, $H = H(u)$. For such a wave, it can be seen from the definition of u (equation (4.1)) that one can take $H \equiv 1$ without loss of generality (see next subsection) so this solution is specified by just two harmonic functions. It is interesting to note that in four dimensions, the null supersymmetric solutions are pp-waves but in five dimensions they are more general plane-fronted waves.

4.2 Changes of coordinate

Note that there is some unfixed gauge freedom in the solution. These correspond to co-ordinate transformations that leave the form of the metric and the field strength invariant. First, equation (4.1) does not define u uniquely. One could instead work with $u' = u'(u)$ and $H' = H du'/du$. This also affects the definition of $\mathbf{x}$: $\mathbf{x}' = \mathbf{x} du/du'$. Next, in defining v by $V = \partial/\partial v$, there is freedom to specify the surface $v = 0$. This corresponds to the coordinate transformation

$$v = v' + g(u, x), \tag{4.28}$$

which has the effect of replacing $\mathbf{a}$ and $\mathcal{F}$ in the solution by

$$\mathbf{a}' = \mathbf{a} - H^{-3}\nabla g, \qquad \mathcal{F}' = \mathcal{F} + 2\partial_u g - 2\mathbf{a}\cdot\nabla g + H^{-3}\left(\nabla g\right)^2. \tag{4.29}$$

This gauge-freedom can be used to impose a gauge condition such as

$$\nabla\cdot\mathbf{a} = 0\ , \tag{4.30}$$

or

$$W_{ii} = 0. \tag{4.31}$$

Finally, it is clear from the definition of the coordinates $\mathbf{x}$ that there is a gauge freedom $\mathbf{x} \to \mathbf{x} - \mathbf{v}(u)$ for an arbitrary u-dependent vector $\mathbf{v}(u)$. These three gauge transformations will be used repeatedly to simplify solutions.

There are other coordinate transformations that do change the form of the solution but are also useful. The coordinates x^i defined above are not arbitrary Cartesian coordinates but are related to the 2-forms $X^{(i)}$ by equation (4.8). Note that these coordinates are defined in terms of covariantly constant 2-forms in exactly the same way as the Cartesian coordinates on the transverse two-space of a four dimensional pp-wave [43]. Any other Cartesian coordinate system on $\mathbb{R}^3$ will be related to these coordinates by a u-dependent rotation and translation:

$$\mathbf{x} = \mathbf{O}(u)\cdot\mathbf{x}' + \mathbf{v}(u), \tag{4.32}$$

where $\mathbf{O}(u)$ is an orthogonal matrix, which we choose to have determinant +1. The effect of such a transformation is to rotate the 2-forms: $X^{(i)} \to \mathbf{O}^{ij}X^{(j)}$. The derivative D_u is invariant under such a transformation. However, the general form of the solution does *not* transform covariantly – the solution as given above is valid only in terms of the preferred coordinates x^i. In particular, the above transformation affect $\mathbf{a}$ differently in the metric and field strength. The new metric will be of the same form of as the old one but with $\mathbf{x}$ replaced by $\mathbf{x}'$ and $\mathbf{a}$ replaced by

$$\mathbf{a}' = \mathbf{O}^{-1}\left(\mathbf{a} + \dot{\mathbf{O}}\cdot\mathbf{x}' + \dot{\mathbf{v}}\right), \tag{4.33}$$

where a dot denotes a u-derivative. The left-hand side of equation (4.21) for the metric function $\mathcal{F}$ is unchanged since $\nabla^2 = \nabla'^2$. Note that if one wants to write the right-hand-side in terms of primed quantities, one should substitute $W = \mathbf{O}W'\mathbf{O}^{-1} + H\dot{\mathbf{O}}\mathbf{O}^{-1}$. The field strength is given by

$$F = -\frac{1}{2\sqrt{3}}\epsilon_{ijk}\left[H^{-2}\nabla'_j\left(H^3 a'_k\right) + H\left(\mathbf{O}^{-1}\dot{\mathbf{O}}\right)_{jk}\right] du \wedge dx'^i - \frac{\sqrt{3}}{4}\epsilon_{ijk}\nabla'_k H dx'^i \wedge dx'^j. \quad (4.34)$$

Note that one can reach $\mathbf{a}' = 0$ (and therefore eliminate $dudx^i$ cross terms from the metric) precisely when $\mathbf{a}$ satisfies Killing's equation $\nabla_{(i}a_{j)} = 0$, i.e, when

$$\mathbf{a} = \mathbf{x} \times \boldsymbol{\omega}(u) + \mathbf{b}(u), \quad (4.35)$$

for some vectors $\boldsymbol{\omega}(u)$ and $\mathbf{b}(u)$. After changing coordinate to eliminate $\mathbf{a}$ from the metric, a $du \wedge dx'^i$ term remains in the field strength so the solution in the new coordinates is not of the same form as the solution in the original coordinates.

4.3 pp-waves

As mentioned above, pp-waves have $H \equiv 1$. Equation (4.18) gives

$$\nabla \times \mathbf{a} = \nabla\phi \quad (4.36)$$

for some function $\phi(u, x)$. The integrability condition for this equation is that ϕ must be harmonic

$$\nabla^2\phi = 0. \quad (4.37)$$

The solution then takes the form

$$\begin{aligned} ds^2 &= \mathcal{F}du^2 + 2dudv - (d\mathbf{x} + \mathbf{a}du)^2 \\ F &= -\frac{1}{2\sqrt{3}}du \wedge d\phi, \qquad \nabla \times \mathbf{a} = \nabla\phi. \end{aligned} \quad (4.38)$$

The function $\mathcal{F}$ is given by solving (4.21).[7] Without loss of generality, we impose the gauge condition $\nabla \cdot \mathbf{a} = 0$, so that (4.21) becomes

$$\nabla^2\mathcal{F} = 2\nabla_{(i}a_{j)}\nabla_{(i}a_{j)} + \frac{1}{3}(\nabla\phi)^2. \quad (4.39)$$

If $\nabla_{(i}a_{j)} = 0$, i.e., if $\mathbf{a} = \mathbf{x} \times \boldsymbol{\omega}(u) + \mathbf{b}(u)$ then one can remove $\mathbf{a}$ from the metric by a coordinate transformation as described above, and $\nabla\phi = -2\boldsymbol{\omega}(u)$ is independent of $\mathbf{x}$. The orthogonal matrix occuring in the coordinate transformation must obey

$$\dot{O}_{ij}(u) = -\epsilon_{ikl}O_{jk}(u)\omega_l(u). \quad (4.40)$$

[7] Some similar ten dimensional pp-wave solutions were given in [44].

In the new coordinates, the solution is

$$
\begin{aligned}
ds^2 &= \mathcal{F}du^2 + 2dudv - d\mathbf{x}'^2 \\
F &= \frac{1}{\sqrt{3}}\omega'_i(u)du \wedge dx'^i, \qquad\qquad \nabla'^2\mathcal{F} = \frac{4}{3}\boldsymbol{\omega}'^2.
\end{aligned}
\tag{4.41}
$$

where $\boldsymbol{\omega}'(u) = \mathbf{O^{-1}}(u)\boldsymbol{\omega}(u)$. Note that $\dot{\boldsymbol{\omega}}' = \mathbf{O}^{-1}(u)\dot{\boldsymbol{\omega}}$ using (4.40). Hence $\boldsymbol{\omega}'$ is independent of u if, and only if, $\boldsymbol{\omega}'$ is (in this case, $\mathbf{O}$ can be taken as a rotation about an axis parallel to $\boldsymbol{\omega}$, giving $\boldsymbol{\omega}' = \boldsymbol{\omega}$). The maximally supersymmetric plane wave solution arising from a Penrose limit (see below) is of this type.

Another special case is that in which $F = 0$, so that the space is a solution of pure gravity, and the Killing spinors are covariantly constant. This case was analysed in [45], where it was shown that the holonomy must be in $\mathbb{R}^3 \subset SO(4,1)$. Setting $F = 0$ gives $\nabla \times \mathbf{a} = 0$, so $\mathbf{a}$ is a gradient and can therefore be set to zero by $v = v' + g(u, \mathbf{x})$. The solution is then

$$
ds^2 = \mathcal{F}du^2 + 2dudv - d\mathbf{x}^2, \qquad \nabla^2\mathcal{F} = 0.
\tag{4.42}
$$

The solution given in [45] (modulo a typo) is related to this by a Euclidean transformation.

In section 3.7, we saw how the timelike class of minimal $N = 2$, $D = 4$ supergravity (with action (3.85)) can be oxidized to give a subset of our timelike class of minimal $N = 1$, $D = 5$ supergravity. We can now do the same for the null class given in [9]; this was done in [42] for the special case of the maximally supersymmetric plane wave solution. Consider a pp-wave with $\mathbf{a} = \mathbf{x} \times \boldsymbol{\omega}(u)$ and consider a coordinate transformation $\mathbf{x} = \mathbf{O}(u)\mathbf{x}'$ with

$$
\dot{O}_{ij}(u) = -\frac{4}{3}\epsilon_{ikl}O_{kj}(u)\omega_l(u).
\tag{4.43}
$$

Note that this is *not* the same coordinate transformation as used to eliminate $\mathbf{a}$ from the metric. In the new coordinates, the solution takes the form

$$
\begin{aligned}
ds^2 &= \mathcal{F}du^2 + 2dudv - \left(d\mathbf{x}' - \frac{1}{3}\mathbf{x}' \times \boldsymbol{\omega}'(u)du\right)^2, \\
F &= \frac{1}{\sqrt{3}}\omega'_i du \wedge dx^{i'},
\end{aligned}
\tag{4.44}
$$

where $\boldsymbol{\omega}'(u) = \mathbf{O^{-1}}(u)\boldsymbol{\omega}(u)$. Now take $\boldsymbol{\omega}' = (\omega'_1(u), 0, \omega'_3(u))$ and let $v = v' - \frac{1}{3}y'(\omega'_3(u)x' - \omega'_1(u)z')$. The solution takes the form

$$
\begin{aligned}
ds^2 &= \mathcal{F}'du^2 + 2dudv' - dx'^2 - dz'^2 - \left(dy' + \frac{2}{3}(\omega'_3x' - \omega'_1z')du\right)^2, \\
F &= \frac{1}{\sqrt{3}}\omega'_i(u)du \wedge dx^{i'},
\end{aligned}
\tag{4.45}
$$

where

$$\mathcal{F}' = \mathcal{F} + \frac{1}{3}(\omega_3' x' - \omega_1' z')^2 - \frac{1}{9}\boldsymbol{\omega}'^2 y'^2 - \frac{2}{3}y'(\dot{\omega}_3' x' - \dot{\omega}_1' z'). \tag{4.46}$$

It is always possible to choose $\mathcal{F}$ obeying (4.21) such that $\mathcal{F}'$ is independent of y'. The solution can then be KK reduced on the Killing vector field $\partial/\partial y'$. In the language of section 3.7, we have $\mathcal{A} = \frac{2}{3}(\omega_3'(u)x' - \omega_1'(u)z')du$ and it can be checked that the consistency conditions for the reduction are obeyed. Explicitly, the four dimensional solution is

$$\begin{aligned} ds^2 &= \mathcal{F}' du^2 + 2du dv' - dx'^2 - dz'^2, \\ F' &= \frac{1}{\sqrt{3}} du \wedge (\omega_1' dx' + \omega_3' dz')\,, \end{aligned} \tag{4.47}$$

where $\mathcal{F}'(u, x', z')$, must obey (using (4.46) and (4.21))

$$\left(\frac{\partial^2}{\partial x'^2} + \frac{\partial^2}{\partial z'^2}\right)\mathcal{F}' = \frac{16}{9}\boldsymbol{\omega}'(u)^2. \tag{4.48}$$

4.4 Black String, Penrose limits and Plane Wave

Probaby the best known solutions in the null class are the static black string solutions [14]. These have $\mathcal{F} = \mathbf{a} = 0$ and $H = H(\mathbf{x})$. A single black string is obtained by choosing the harmonic function H to have a single centre:

$$\begin{aligned} ds^2 &= H^{-1}(2du dv) - H^2(d\mathbf{x})^2 \\ F &= -\frac{\sqrt{3}}{4}\epsilon_{ijk}\nabla_k H dx^i \wedge dx^j, \qquad H = 1 + \frac{R}{2r} \end{aligned} \tag{4.49}$$

The tension and magnetic charge per unit length of the string are both proportional to R.

The near horizon limit of this solution is obtained by taking $r \to 0$ and hence dropping the 1 in the harmonic function. One then obtains $AdS_3 \times S^2$, which is maximally supersymmetric. This can be written in global coordinates as

$$\begin{aligned} ds^2 &= R^2[\cosh^2\rho dt^2 - d\rho^2 - \sinh^2\rho d\psi^2 - \frac{1}{4}(d\theta^2 + \cos^2\theta d\phi^2)] \\ F &= \frac{\sqrt{3}R}{4}\cos\theta d\theta \wedge d\phi \end{aligned} \tag{4.50}$$

where $0 \leq \psi, \phi < 2\pi$ and $0 \leq \theta \leq \pi$.

The maximally supersymmetric plane-wave solution presented in [21] can be obtained as a Penrose limit. Explicitly if we introduce the new coordinates

$$\begin{aligned} u &= t + \frac{\phi}{2}, \qquad v = \frac{R^2}{2}(t - \frac{\phi}{2}), \\ \rho &= \frac{r}{R}, \qquad \theta = -\frac{2z}{R} \end{aligned} \tag{4.51}$$

and then take the limit $R \to \infty$ we get

$$
\begin{aligned}
ds^2 &= 2dudv + (z^2 + \frac{r^2}{4})du^2 - (dr^2 + r^2 d\psi^2 + dz^2) \\
F &= \frac{\sqrt{3}}{2} du \wedge dz
\end{aligned}
\tag{4.52}
$$

Note that this solution is of the form (4.41) with $\boldsymbol{\omega} = (0, 0, 3/2)$. Note also that there is a similar Penrose limit of $AdS_2 \times S^3$, the near horizon limit of the electric black hole solution, that gives the same maximal supersymmetric plane-wave solution.

5 Maximally supersymmetric solutions

5.1 Introduction

The above results for the timelike and null cases show that Killing spinors always come in pairs (of Dirac spinors) obeying the same projection (3.11). Therefore the solutions all preserve at least half of the supersymmetry. It is natural to ask which solutions preserve more than half of the supersymmetry.

A pair spans a two dimensional subspace of spinors. Every spinor in this subspace gives rise to the same function f and Killing vector V. Let ϵ_1 and ϵ_2 be a pair. If there is an extra linearly independent Killing spinor ϵ_3 then it must also have a partner ϵ_4. Let f', V' denote the function and Killing vector that arises from this second pair.

Consider first the case in which ϵ_4 is not linearly independent: $\epsilon_4 = \alpha\epsilon_1 + \beta\epsilon_2 + \gamma\epsilon_3$, for some functions α, β, γ. Since each ϵ_i is Killing and, by assumption, ϵ_1, ϵ_2 and ϵ_3 are linearly independent, we conclude that α, β, γ are actually constants. In addition $\epsilon' \equiv \alpha\epsilon_1 + \beta\epsilon_2$ is a Killing spinor obeying the same projection as ϵ_4, i.e., it forms a pair with ϵ_3. It follows that ϵ' gives rise to the function f' and Killing vector V'. But, being a linear combination of ϵ_1 and ϵ_2, ϵ' must give rise to the Killing vector V. Hence we must have $V = V'$ and similarly $f = f'$ (at least up to a positive constant of proportionality). However, this implies that ϵ_4 obeys precisely the same projection (equation (2.13)) as ϵ_1 and ϵ_2, which contradicts the linear independence of ϵ_3. Hence ϵ_4 must be linearly independent of ϵ_1, ϵ_2 and ϵ_3. So if a solution preserves more than 1/2 supersymmetry then it must preserve all supersymmetry.

The goal of this section is to identify those solutions preserving all supersymmetry. This can be done by examining the integrability conditions. If there are four independent Killing spinors then it is easy to argue that there must exist an open set $\mathcal{U}$ in which these Killing spinors are pointwise linearly independent and hence form a basis for all spinors. The integrability conditions are algebraic and must therefore hold for an arbitrary spinor in $\mathcal{U}$. This yields an identity of the form

$$X_{\alpha\beta\gamma}\gamma^{\gamma} + Y_{\alpha\beta\gamma\delta}\gamma^{\gamma\delta} = 0, \tag{5.1}$$

where X and Y are tensors formed from the field strength and Riemann tensor. By analytic continuation, this must hold everywhere, not just in $\mathcal{U}$. This expression can only be valid if X and Y vanish separately. Our strategy will be to examine what further restrictions this gives for the solutions classified above. Rather than computing X and Y directly from the Riemann tensor and field strength, it turns out to be simpler to write out the Killing spinor equation in components and rederive the integrability conditions component by component, which will clearly give identical results.

5.2 Maximal null supersymmetry

In the above analysis, we showed that the null solutions admit Killing spinors obeying the projection $\gamma^+\epsilon = 0$. We now want to find the maximally supersymmetric solutions and must therefore relax this condition. Any spinor ϵ can be written as

$$\epsilon = \epsilon_+ + \epsilon_-, \tag{5.2}$$

where $\epsilon_+ = (1/2)\gamma^+\gamma^-\epsilon$ and $\epsilon_- = (1/2)\gamma^-\gamma^+\epsilon$. Substituting the known form of the null solution into the Killing spinor equation yields the following components:

$$\partial_v \epsilon_+ + \frac{1}{2}H^{-2}\nabla_i H\gamma^i\gamma^+\epsilon_- = 0, \tag{5.3}$$

$$\partial_v \epsilon_- = 0, \tag{5.4}$$

$$\nabla_i\epsilon_+ - \frac{1}{2}H\left[\frac{1}{3}H\nabla_{[i}a_{j]} + W_{(ij)}\right]\gamma^j\gamma^+\epsilon_- - \frac{1}{6}H^2\epsilon_{ijk}\nabla_j a_k\gamma^+\epsilon_- = 0, \tag{5.5}$$

$$\nabla_i\epsilon_- - H^{-1}\epsilon_{ijk}\nabla_j H\gamma^k\epsilon_- = 0, \tag{5.6}$$

$$\left(\partial_u - a^i\nabla_i\right)\epsilon_+ - \frac{1}{4}\nabla_i\left(\mathcal{F}H^{-1}\right)\gamma^i\gamma^+\epsilon_- = 0, \tag{5.7}$$

$$\left(\partial_u - a^i\nabla_i\right)\epsilon_- + \frac{1}{3}\epsilon_{ijk}\nabla_j a_k\gamma^i\epsilon_- = 0. \tag{5.8}$$

Consider the integrability condition for (5.3) and (5.5). Using (5.4) and (5.6), this reduces to

$$\left[H^{-2}\nabla_i\nabla_j H - 3H^{-3}\nabla_i H\nabla_j H + H^{-3}\left(\nabla H\right)^2\delta_{ij}\right]\gamma^j\gamma^+\epsilon_- = 0. \tag{5.9}$$

We now apply the argument outlined above: ϵ_- can be replaced by ϵ in this expression, and we are demanding that there exist eight independent solutions. This can only be true if the expression within square brackets vanishes, which is equivalent to

$$\nabla_i\nabla_j H^{-2} = 2\left(\nabla H^{-1}\right)^2\delta_{ij}. \tag{5.10}$$

This equation is easy to solve (e.g. first consider the components with $i \neq j$) and has solutions $H = H(u)$ or $H^{-2} = f(u)^{-2} (\mathbf{x} + \mathbf{b}(u))^2$. In the latter case, one can exploit the unfixed coordinate freedom $\mathbf{x} \to \mathbf{x} - \mathbf{c}(u)$ to set $\mathbf{b} = 0$. In the former case, it was argued above that one can change coordinates so that $H \equiv 1$. Hence

$$H \equiv 1, \qquad \text{or} \qquad H = \frac{f(u)}{r}, \tag{5.11}$$

where $r \equiv \sqrt{\mathbf{x}^2}$.

Consider first the case $H = f(u)/r$. The integrability condition of (5.7) and (5.3) implies (using (5.4) and (5.8))

$$\partial_u \left(H^{-2} \nabla_i H \right) - \frac{H^{-5}}{3} \nabla_j H \left[\nabla_i \left(H^3 a_j \right) - \nabla_j \left(H^3 a_i \right) \right] = 0, \tag{5.12}$$

and

$$\nabla H \cdot \nabla \times \mathbf{a} = 0. \tag{5.13}$$

Multiplying the first equation by $\nabla_i H$ leads immediately to $\partial_u f = 0$ and hence

$$H = \frac{R}{2r}, \tag{5.14}$$

for some constant R, and a factor of 2 is inserted for convenience. Equations (5.12) and (5.13) then imply

$$\nabla \times \left(H^3 \mathbf{a} \right) = 0, \tag{5.15}$$

which implies that $\mathbf{a}$ can be gauged away by a transformation of the form $v \to v - g(u, \mathbf{x})$. Hence we can assume $\mathbf{a} = 0$. Integrability of (5.5) and (5.7) then implies

$$\nabla \mathcal{F} \times \nabla H = 0, \tag{5.16}$$

so $\mathcal{F} = \mathcal{F}(u, r)$. Equation (4.21) implies that $\mathcal{F}$ is harmonic, hence

$$\mathcal{F} = \frac{f_1(u)}{r} + f_2(u). \tag{5.17}$$

Finally, by considering a coordinate transformation of the form $v = v' + P_1(u)/r' + P_2(u)$, $r = r' P_3(u)$, $u = P_4(u')$ for appropriate choices of the functions P_i one can bring the solution to the form

$$ds^2 = 4\frac{r'}{R} du' dv' - \frac{R^2}{4} \frac{dr'^2}{r'^2} - \frac{R^2}{4} d\Omega_2^2, \qquad F = \frac{\sqrt{3}}{4} R \mathrm{vol}(S^2), \tag{5.18}$$

which is clearly the maximally supersymmetric $AdS_3 \times S^2$ solution (4.50). If we drop the primes on the coordinates then this solution is simply given by setting $\mathcal{F} = \mathbf{a} = 0$, $H = R/2r$ and, continuing to use the frame (4.11), the Killing spinors are easily found to be

$$\epsilon_- = \hat{x}^i \gamma^i \eta_-, \tag{5.19}$$

$$\epsilon_+ = \eta_+ + \frac{v}{R}\gamma^+\eta_-, \tag{5.20}$$

where $\hat{\mathbf{x}} = \mathbf{x}/r$ and $\eta_\pm$ are arbitrary constant spinors obeying $\gamma^\pm\eta_\pm = 0$.

Now consider the case $H \equiv 1$. Integrability of (5.6) and (5.8) gives

$$\nabla \times \mathbf{a} = -2\boldsymbol{\omega}(u), \tag{5.21}$$

where $\boldsymbol{\omega}(u)$ is an arbitrary u-dependent vector and hence

$$\mathbf{a} = \mathbf{x} \times \boldsymbol{\omega}(u), \tag{5.22}$$

where an arbitrary gradient can be removed by a shift in v. Integrability of (5.5) and (5.7) then gives

$$\boldsymbol{\omega} = \text{constant}, \tag{5.23}$$

and

$$\nabla_i\nabla_j\mathcal{F} = \frac{2}{9}\left(\delta_{ij}\boldsymbol{\omega}^2 + 3\omega_i\omega_j\right), \tag{5.24}$$

with solution

$$\mathcal{F} = \frac{1}{9}\left(\boldsymbol{\omega}^2\mathbf{x}^2 + 3\left(\boldsymbol{\omega}\cdot\mathbf{x}\right)^2\right) + \boldsymbol{\alpha}(u)\cdot\mathbf{x} + \beta(u). \tag{5.25}$$

The arbitrary functions $\boldsymbol{\alpha}(u)$ and $\beta(u)$ can be removed by a combined transformation of the form $\mathbf{x} \to \mathbf{x} - \boldsymbol{\gamma}(u)$ and $v \to v - \boldsymbol{\lambda}(u)\cdot\mathbf{x} - \delta(u)$. Hence the final form of the solution is

$$\begin{aligned} ds^2 &= \frac{1}{9}\left(\boldsymbol{\omega}^2\mathbf{x}^2 + 3\left(\boldsymbol{\omega}\cdot\mathbf{x}\right)^2\right)du^2 + 2dudv - \left(d\mathbf{x} + \mathbf{x}\times\boldsymbol{\omega}du\right)^2, \\ F &= \frac{1}{\sqrt{3}}\omega_i du\wedge dx^i, \end{aligned} \tag{5.26}$$

where $\boldsymbol{\omega}$ is an arbitrary constant 3-vector. It is now easy to solve for the Killing spinors:

$$\epsilon_- = \left[\cos\left(2\omega u/3\right) + \hat{\omega}_i\gamma^i\sin\left(2\omega u/3\right)\right]\eta_-, \tag{5.27}$$

$$\epsilon_+ = \eta_+ + \frac{1}{6}\left(\mathbf{x}\wedge\boldsymbol{\omega}\right)_i\gamma^i\gamma^+\epsilon_- - \frac{1}{3}\mathbf{x}\cdot\boldsymbol{\omega}\gamma^+\epsilon_-, \tag{5.28}$$

where $\omega = |\boldsymbol{\omega}|$, $\hat{\boldsymbol{\omega}} = \boldsymbol{\omega}/\omega$, and $\eta_\pm$ are arbitrary constant spinors obeying the projections $\gamma^\pm\eta_\pm = 0$. In the coordinate system of equation (4.41), the solution is (using $\boldsymbol{\omega}' = \boldsymbol{\omega}$)

$$ds^2 = \frac{1}{9}\left(\boldsymbol{\omega}^2\mathbf{x}'^2 + 3\left(\boldsymbol{\omega}\cdot\mathbf{x}'\right)^2\right)du^2 + 2dudv - d\mathbf{x}'^2, \qquad F = \frac{1}{\sqrt{3}}\omega_i du\wedge dx'^i. \tag{5.29}$$

We thus conclude that the only maximally supersymmetric solutions in the null class are $AdS_3 \times S^2$ and the maximally supersymmetric plane wave. It turns out that both of these solutions also belong to the timelike class. In other words, some of the Killing spinors correspond to a null Killing vector but others correspond to a timelike Killing vector. This is easy to see using the explicit expressions for the Killing spinors given above. For the plane wave, take a Killing spinor with $\eta_+ = 0$ and imagine repeating the analysis of this paper with this spinor as the fiducial spinor. One then obtains

$$f = -\frac{\sqrt{2}}{3}\mathbf{x} \cdot \boldsymbol{\omega} \eta_-^\dagger \eta_-, \tag{5.30}$$

which is clearly non-zero (although we need to restrict to $\mathbf{x} \cdot \boldsymbol{\omega} < 0$ for $f > 0$). Similarly, for the $AdS_3 \times S^2$ solution one obtains from the general Killing spinor

$$f = -\frac{1}{\sqrt{2}}\hat{x}^i \Re\left(\eta_-^\dagger \gamma^i \gamma^- \eta_+\right), \tag{5.31}$$

where $\Re$ denotes the real part, which is also non-zero in general. These solutions can be cast into the timelike form of section 3 as follows.

For the plane wave, a Killing spinor with $\eta_+ = 0$ gives the Killing vector

$$V = \frac{\partial}{\partial u} - \frac{4}{3}a_i \frac{\partial}{\partial x^i}, \tag{5.32}$$

where we have normalized so that $\eta_-^\dagger \eta_- = \sqrt{2}$. We want to write the solution in the timelike form, so we need to choose new coordinates (t, y^m) such that $V = \partial/\partial t$. A natural guess is to choose new coordinate $(t, x^5, \mathbf{x}')$ where $t = u$, $x^5 = v$ and

$$\frac{\partial x^i}{\partial t} = -\frac{4}{3}a_i. \tag{5.33}$$

A solution is to take $\mathbf{x} = \mathbf{O}(t)\mathbf{x}'$ where $\mathbf{O}(t) = \exp(\mathbf{A}t)$ is orthogonal and the antisymmetric matrix $\mathbf{A}$ is given by

$$A_{ij} = -\frac{4}{3}\epsilon_{ijk}\omega_k. \tag{5.34}$$

Note that the same coordinate transformation was used above in the dimensional reduction of the null class. In these new coordinates, the solution takes the form

$$\begin{aligned} vds^2 &= f^2\left(dt + \omega\right)^2 - f^{-1}\left[f^{-1}\left(dx^5 + \boldsymbol{\chi} \cdot d\mathbf{x}'\right)^2 + f d\mathbf{x}'^2\right], \\ F &= \frac{1}{\sqrt{3}}\omega_i dt \wedge dx'^i, \end{aligned} \tag{5.35}$$

where

$$f = -\frac{2}{3}\mathbf{x}' \cdot \boldsymbol{\omega}, \qquad \boldsymbol{\chi} = \frac{1}{3}\mathbf{x}' \times \boldsymbol{\omega}, \qquad \omega = f^{-2}\left(dx^5 + \boldsymbol{\chi} \cdot d\mathbf{x}'\right). \tag{5.36}$$

The solution is now written in the timelike form. The base space is a Gibbons-Hawking space with a linear harmonic function $H = f$, corresponding to a constant density planar distribution of Taub-NUT instantons. In the language of our general analysis of solutions with a Gibbons-Hawking base space, this solution has $K = 1$ and $L = M = \boldsymbol{\omega} = 0$. Note that K/H is not constant, so this solution has $G^+ \neq 0$.

For $AdS_3 \times S^2$, the Killing vector V constructed from the general Killing spinor turns out to be rather complicated, but it is possible to proceed by trial and error. First, note that we can rotate S^2 and normalize the spinor so that $f = \cos\theta$. Now the solution can be massaged into timelike form by writing AdS_3 in a form with $\mathcal{F} = R/2r$ (as discussed above):

$$\begin{aligned} ds^2 &= du^2 + 4\frac{r}{R}dudv - \frac{R^2}{4r^2}dr^2 - \frac{R^2}{4}\left(d\theta^2 + \sin^2\theta d\phi^2\right), \\ F &= \frac{\sqrt{3}}{4}R\sin\theta d\theta \wedge d\phi. \end{aligned} \tag{5.37}$$

Letting $u = t$, $v = x^5$, $\phi = \phi' - 2t/R$, the solution can be written

$$\begin{aligned} ds^2 &= f^2(dt+\omega)^2 - f^{-1}\left[H\left(dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi'^2\right) + H^{-1}\left(dx^5 + \frac{R^2\sin^2\theta}{4r}d\phi'\right)^2\right], \\ F &= \frac{\sqrt{3}}{2}\sin\theta\left(dt\wedge d\theta + \frac{R}{2}d\theta\wedge d\phi'\right), \end{aligned} \tag{5.38}$$

where

$$f = \cos\theta, \qquad \omega = \frac{2r}{R\cos^2\theta}\left(dx^5 + \frac{R^2\sin^2\theta}{4r}d\phi'\right), \qquad H = \frac{R^2\cos\theta}{4r^2}. \tag{5.39}$$

The base space is a Gibbons-Hawking solution with a dipole source for H. K has a monopole source: $K = R/2r$ and $L = M = \boldsymbol{\omega} = 0$. Once again, K/H is not constant so this solution has $G^+ \neq 0$.

It is noteworthy that both of these regular maximally supersymmetric solutions can be cast in the time-like form using singular hyper-Kähler base spaces. Note also that for both solutions, the null Killing vector field $\partial/\partial v$ coincides with the triholomorphic Killing vector field $\partial/\partial x^5$. Both solutions are special cases of the class of timelike solutions given in section 3.7 with $L = M = \boldsymbol{\omega} = 0$ (so $f^{-1} = K^2/H$ and $\omega_5 = (K/H)f^{-1}$) for which $\partial/\partial x^5$ is null.

It was remarked in [42] that $AdS_3 \times S^2$ can be obtained as a limit of the maximally supersymmetric near horizon geometry of the BMPV black hole discussed in [16]. It is interesting to see how this works in our framework. The near horizon geometry of BMPV has a flat base space and $G^+ = 0$. In Gibbons-Hawking form, H has a monopole source and f^{-1} and ω_5 are proportional to H. More generally, consider a solution with Gibbons-Hawking base, $f^{-1} = H$ and $\omega_5 = cH$ for some constant c:

$$\begin{aligned} ds^2 &= H^{-2}dt^2 + 2cH^{-1}dt\sigma - (1-c^2)\sigma^2 - H^2 d\mathbf{x}^2, \qquad \sigma = dx^5 + \chi_i dx^i, \\ F &= \frac{\sqrt{3}}{2}\left(-dt\wedge d(H^{-1}) + \frac{c}{2}\epsilon_{ijk}\nabla_k H dx^i \wedge dx^j\right), \end{aligned} \tag{5.40}$$

For $c^2 < 1$ one can KK reduce on $\partial/\partial x^5$ as described in section 3.7 (after rescaling co-ordinates). If $c^2 = 1$ then $\partial/\partial x^5$ is null and this reduction is no longer possible (although the metric remains non-degenerate). However, one can introduce new coordinates

$$t' = \frac{t}{\sqrt{1-c^2}}, \qquad x^{5'} = \sqrt{1-c^2}\,x^5, \tag{5.41}$$

and then take the limit $c^2 \to 1$ with t' and $x^{5'}$ held fixed. This is not the same as setting $c^2 = 1$ in the original solution. For $c \to 1$ we get

$$ds^2 = H^{-1}\left(-Hdx^{5'^2} + 2dt'dx^{5'}\right) - H^2 d\mathbf{x}^2, \qquad F = \frac{\sqrt{3}}{4}\epsilon_{ijk}\nabla_k H dx^i \wedge dx^j, \tag{5.42}$$

which is a solution belonging to the null class with $u = x^{5'}$, $v = t'$, $\mathcal{F} = -H$ and $\mathbf{a} = 0$.[8] This limit must therefore involve a boost. If H has a monopole source then the above procedure takes the near horizon geometry of BMPV to $AdS_3 \times S^2$ written in null form.

5.3 Maximal Timelike Supersymmetry

We now determine the maximally supersymmetric solutions which are associated with a timelike killing vector by analysing the Killing spinor equations in the timelike background described in section 3. We decompose the Killing spinor as $\epsilon = \epsilon^- + f^{\frac{1}{2}}\epsilon^+$ where $\gamma^0\epsilon^\pm = \pm\epsilon^\pm$. Then the Killing spinor equations may be written as

$$\partial_t \epsilon^+ = -\gamma^i \nabla_i f \epsilon^- \tag{5.43}$$

$$\partial_t \epsilon^- = \frac{f^2}{6}\gamma^{ij} {G^+}_{ij}\epsilon^- \tag{5.44}$$

$$\nabla_i \epsilon^+ + \omega_i \gamma^j \nabla_j f \epsilon^- - \frac{1}{3}{G^+}_{ij}\gamma^j \epsilon^- - {G^-}_{ij}\gamma^j\epsilon^- = 0 \tag{5.45}$$

and

$$\nabla_i \epsilon^- - \frac{f^2}{6}\omega_i {G^+}_{jk}\gamma^{jk}\epsilon^- + \frac{1}{2}f^{-1}\delta_{ij}\nabla_k f \gamma^{jk}\epsilon^- + \frac{1}{2}f^{-1}\nabla_i f \epsilon^- = 0 \tag{5.46}$$

In the above equations, all spatial indices are with respect to an orthonormal basis with respect to the base space metric h, and ∇ is the covariant derivative with respect to h.

To proceed, we evaluate the integrability conditions of these equations. First, for maximal supersymmetry, the integrability condition of (5.44) and (5.46) gives

$$\nabla_i(f^2 {G^+}_{pq}) - f\delta_{ip}\nabla_k f \delta^{k\ell}{G^+}_{lq} + f\delta_{iq}\nabla_k f\delta^{k\ell}{G^+}_{lp} + f\nabla_p f {G^+}_{iq} - f\nabla_q f {G^+}_{ip} = 0 \tag{5.47}$$

[8] The observant reader will notice a discrepancy in the sign of the field strength between this equation and (4.16). This arises because the null solutions have no preferred orientation. The sign can be fixed by $\mathbf{x} \to -\mathbf{x}$.

The integrability condition between (5.43) and (5.45) is

$$-\nabla_i \nabla_j f + \frac{1}{2f}\delta_{ij}\delta^{pq}\nabla_p f \nabla_q f + \frac{2}{3}f^2(\frac{1}{3}G^+{}_{ip} + G^-{}_{ip})\delta^{pq}G^+{}_{qj} = 0 \tag{5.48}$$

The integrability condition of (5.46) is

$$C_{ijpq} \;=\; f[-\frac{2}{3}G^+{}_{ij}G^+{}_{pq} + \frac{1}{18}G^+{}_{mn}G^{+mn}(\delta_{ip}\delta_{jq} - \delta_{jp}\delta_{iq} + \epsilon_{ijpq})] \tag{5.49}$$

where C is the Weyl tensor, and the integrability condition of (5.45) is

$$\begin{aligned} -f\nabla_\ell G^-{}_{ij} + (\nabla_i f G^-{}_{j\ell} - \nabla_j f G^-{}_{i\ell} - \nabla_\ell f G^-{}_{ij}) + (\delta_{j\ell}G^-{}_{im}\delta^{mn}\nabla_n f - \delta_{i\ell}G^-{}_{jm}\delta^{mn}\nabla_n f) \\ +\frac{1}{3}(\nabla_\ell f G^+{}_{ij} + \nabla_i f G^+{}_{j\ell} - \nabla_j f G^+{}_{i\ell}) - \frac{1}{3}(\delta_{j\ell}G^+{}_{im}\delta^{mn}\nabla_n f - \delta_{i\ell}G^+{}_{jm}\delta^{mn}\nabla_n f) = 0 \end{aligned} \tag{5.50}$$

Let us first investigate the special case $G^+ = 0$. From (5.49) we immediately conclude that the base space is flat. In fact, the square of (5.49) relates the square of the Weyl tensor of the base space to the square of G^+ and hence a maximally supersymmetric solution has flat base space if, and only if, $G^+ = 0$. Solving (5.48) we find that the two possible solutions are:

$$f \;=\; \alpha \; or, \tag{5.51}$$

$$f \;=\; \frac{\alpha}{2}x^m x^m = \frac{\alpha}{2}r^2 \tag{5.52}$$

for $\alpha > 0$ constant.

We consider first the case $f = \alpha$. Then (5.50) implies that G^- is covariantly constant. Now any anti-self-dual two form can be expanded in terms of the standard anti-self-dual complex structures $J^{(i)} = \frac{1}{4}d[r^2\sigma^i_L]$ on $\mathbb{R}^4$ as $G^- = \lambda^i J^{(i)}$ and we deduce that the λ^i are constants; so

$$\omega = \frac{\lambda^i r^2}{4\alpha}\sigma^i_L \tag{5.53}$$

The five-dimensional metric is given by

$$ds^2 = \alpha^2(dt + \frac{\lambda^i r^2}{4\alpha}\sigma^i_L)^2 - \alpha^{-1}[dr^2 + r^2 d\Omega_3{}^2] \tag{5.54}$$

and is the maximally supersymmetric Gödel type solution investigated previously.

Let us now consider the case $f = \frac{\alpha}{2}r^2$. We introduce a new basis of anti-self-dual two forms $Q^{(i)} = d[r^{-2}\sigma^i_R]$. Then writing $G^- = \lambda^i r^2 Q^{(i)}$ we find on substituting into (5.50) that the λ^i must be constant. Hence

$$\omega = \frac{2}{\alpha r^2}\lambda^i \sigma^i_R \tag{5.55}$$

The five dimensional spacetime geometry is given by

$$ds^2 = \frac{\alpha^2}{4}r^4(dt + \frac{2}{\alpha r^2}\lambda^i\sigma^i_R)^2 - \frac{2}{\alpha r^2}[dr^2 + r^2 d\Omega_3{}^2] \tag{5.56}$$

This geometry is the near-horizon geometry of the rotating BMPV five-dimensional black hole which was shown to be maximally supersymmetric in [16]. Setting $\lambda^i = 0$ gives $AdS_2 \times S^3$.

5.3.1 Maximal supersymmetry with $G^+ \neq 0$

If $G^+ \neq 0$ then the above equations are much more complicated. Before solving these equations, it is useful to consider two examples of maximally supersymmetric solutions with $G^+ \neq 0$. Both examples use a singular hyper-Kähler base, the first a singular version of Eguchi-Hanson and the second negative mass Taub-NUT. Surprisingly, both examples are related by co-ordinate transformations to time-like solutions built from a flat base space with $G^+ = 0$. This emphasizes the point that G^+ is defined with respect to a particular four dimensional base space, which in turn is defined by a timelike Killing vector constructed from a Killing spinor. For solutions with maximal supersymmetry it is possible that the extra Killing spinors give rise to a different timelike Killing vector and hence a different base space, and in this case there will then be no simple relation between the old and new G^+.

Consider first the singular Eguchi-Hanson Solution with base metric on B given by

$$ds^2 = W^{-1}dr^2 + \frac{r^2}{4}((\sigma_L^1)^2 + (\sigma_L^2)^2) + \frac{r^2}{4}W(\sigma_L^3)^2 \tag{5.57}$$

where $W = 1 + \frac{b^4}{r^4}$. Take the solution given by (3.53) with $\delta = \lambda = \gamma = 0$:

$$\begin{aligned} f^{-1} &= \frac{\chi^2}{9b^4r^2}, \qquad \omega = -\frac{\chi^3}{54b^4r^4}\sigma_L^3, \\ G^+ &= -\frac{\chi}{4}d(r^{-2}\sigma_L^3), \qquad G^- = \frac{\chi}{6r^3}(dr \wedge \sigma_L^3 - \frac{r}{2}\sigma_L^1 \wedge \sigma_L^2). \end{aligned} \tag{5.58}$$

It is straightforward to show that all of the integrability conditions given above are satisfied. The metric is

$$ds^2 = \left(\frac{3b^2}{\chi}\right)^4 r^4 dt^2 - \frac{3b^4}{\chi}dt\sigma_L^3 - \frac{\chi^2}{9b^4r^2}\left(1 + \frac{b^4}{r^4}\right)^{-1} dr^2 - \frac{\chi^2}{36b^4}\left[(\sigma_L^1)^2 + (\sigma_L^2)^2 + (\sigma_L^3)^2\right]. \tag{5.59}$$

Now perform a coordinate transformation

$$dv = dt + F(r)dr, \qquad d\phi' = d\phi + G(r)dr \Rightarrow \sigma_L^{3\prime} = \sigma_L^3 + G(r)dr, \tag{5.60}$$

with F and G chosen so that the coefficients of dr^2 and $dr(\sigma_L^3)'$ vanish. The new metric is

$$ds^2 = \left(\frac{3b^3}{\chi}\right)^4 \left(1 + \frac{r^4}{b^4}\right) dv^2 - \frac{6b^2r}{\chi}dvdr - \frac{\chi^2}{36b^4}\left[(\sigma_L^1)^2 + (\sigma_L^2)^2 + \left(\sigma_L^{3\prime} + \frac{54b^8}{\chi^3}dv\right)^2\right]. \tag{5.61}$$

Finally, let

$$\phi'' = \phi' + \frac{54b^8}{\chi^3}v, \qquad v' = \left(\frac{3b^3}{\chi}\right)^2 v, \qquad \rho = \frac{\chi r^2}{6b^4}. \tag{5.62}$$

The metric is now

$$\left[1 + \left(\frac{6b^2}{\chi}\right)^2 \rho^2\right] dv'^2 - 2dv'd\rho - \frac{\chi^2}{36b^4}\left[d\theta^2 + \sin^2\theta d\psi^2 + (d\phi'' + \cos\theta d\psi)^2\right]. \tag{5.63}$$

This is clearly $AdS_2 \times S^3$ where the radius of the AdS_2 is given by $\chi/(6b^2)$ and the radius of the S^3 by $\chi/(3b^2)$. Note that r corresponds to the *global* radial coordinate ρ, with $r = 0$ the origin of AdS_2. This contrasts with the description of $AdS_2 \times S^3$ with a flat base space, for which r is the horospherical "radial" coordinate.

Our next example has negative mass Taub-NUT as its base space. The solution is given by setting $\gamma = 0$, $a = -b < 0$ in (3.60). Explicitly, the base space metric is

$$ds^2 = \frac{(r-b)}{(r+b)}dr^2 + (r^2-b^2)((\sigma^1_R)^2 + (\sigma^2_R)^2) + 4b^2\frac{(r-b)}{(r-b)}(\sigma^3_R)^2 \tag{5.64}$$

and the solution is given by

$$\begin{aligned} f^{-1} &= \frac{2\chi^2}{9b(r-b)}, \qquad \omega = \frac{\chi^3}{27b^2}\frac{(r-5b)(r+b)}{(r-b)^2}\sigma^3_R \\ G^+ &= \chi d[\frac{(r+b)}{(r-b)}\sigma^3_R], \qquad G^- = -\frac{\chi}{6b}\frac{(r+b)}{(r-b)^2}[2bdr\wedge\sigma^3_R + (r^2-b^2)\sigma^1_R\wedge\sigma^2_R] \ . \end{aligned} \tag{5.65}$$

It is straightforward to show that these expressions satisfy the integrability conditions. Surprisingly this solution is just the maximally supersymmetric Gödel type solution. To see this we first note that in going from positive mass parameter Taub-NUT to negative mass Taub-NUT there is a change of orientation. Hence it is natural to work with right invariant one-forms rather than left-invariant one-forms. This is simply achieved by interchanging the coordinates ϕ and ψ. If we do this then the metric becomes

$$\begin{aligned} ds^2 &= \frac{81b^2}{4\chi^4}(r-b)^2dt^2 - \frac{3}{2\chi}(r-5b)(r+b)dt\sigma^3_L - \frac{2\chi^2}{9b(r+b)}dr^2 \\ &- \frac{2\chi^2}{9b}(r+b)((\sigma^1_L)^2 + (\sigma^2_L)^2) + \frac{\chi^2}{36b^2}(r+b)(r-7b)(\sigma^3_L)^2 \end{aligned} \tag{5.66}$$

Let $\phi' = \phi - (3/\chi)^3 b^2 t$ and $r = -b + \rho^2/(8b)$. The metric becomes

$$\begin{aligned} ds^2 &= \left(\frac{9b^2}{\chi^2}\right)^2\left[dt - \left(\frac{\chi}{3}\right)^3\left(\frac{\rho^2}{16b^4}\right)(d\phi' + \cos\theta d\psi)\right]^2 \\ &- \left(\frac{9b^2}{\chi^2}\right)^{-1}\left[d\rho^2 + \frac{\rho^2}{4}\left(d\theta^2 + \sin^2\theta d\psi^2 + (d\phi' + \cos\theta d\psi)^2\right)\right], \end{aligned} \tag{5.67}$$

which, after rescaling t and ρ, is the generalized Gödel solution (3.43) with $\gamma = -3/16\chi$.

To proceed with finding the maximally supersymmetric timelike solutions with $G^+ \neq 0$ it is convenient to prove the

Proposition. The hyper-Kähler base space B of the maximally supersymmetric solutions is Gibbons-Hawking. Moreover, the tri-holomorphic Killing vector is a Killing vector of the five-dimensional solution.

Proof.

Using (5.47), (5.48) and (5.50) it follows that

$$K^i = f(G^{+ij} - 3G^{-ij})\nabla_j f \tag{5.68}$$

satisfies $\nabla_{(i}K_{j)} = 0$ and $\mathcal{L}_K X^{(i)} = 0$. So if $K \neq 0$, as for the negative mass Taub-NUT example presented in the previous section, it follows that the base space is Gibbons-Hawking. Moreover, it is clear that $\mathcal{L}_K f = 0$. In order to show that this solution falls into the classification of Gibbons-Hawking solutions presented previously, we also require $\mathcal{L}_K \omega = 0$. In fact, it suffices to show locally that $\mathcal{L}_K d\omega = 0$. To do this, we note that on contracting (5.47) with $f^2 G^{+pq}$ we find $z^2 = f^4 G^+{}_{ij} G^{+ij}$ is constant; $z \neq 0$. Then it is straightforward to see from the integrability constraints that

$$f^{-1}K^j(G^+{}_{ij} + G^-{}_{ij}) = \nabla_i(\frac{z^2}{12}f^{-3} - \frac{3}{4}fG^-{}_{mn}G^{-mn}) \tag{5.69}$$

and hence $d(i_K d\omega) = 0$.

It is however also possible that $K = 0$, as it is for the singular Eguchi-Hanson example discussed in the previous section. To proceed in this case, we note that $K = 0$ together with (5.50) implies that $f^2 G^-$ is covariantly constant. It is then convenient to define

$$\hat{K}_i = f^2 G^-{}_{ij}\nabla^j f \ . \tag{5.70}$$

Note that if $\hat{K}$ vanishes, or f is constant, then the base space must be flat. Hence we shall consider $\hat{K} \neq 0$. It is straightforward to show that $\hat{K}$ is a Killing vector. Furthermore, without loss of generality we have $f^2 G^- = \frac{z}{6}X^{(1)}$. Next, note that (5.48) implies that

$$\frac{1}{2}f^{-1}\nabla^i f \nabla_i f = \frac{z^2}{18}f^{-2} + \alpha \tag{5.71}$$

for constant α. Hence we find that

$$\nabla_i \hat{K}_j = -\frac{\alpha z}{6}X^{(1)}_{ij} + \frac{z^2}{54}G^+{}_{ij} \tag{5.72}$$

Furthermore, when $K = 0$ the integrability conditions imply the following useful identities:

$$\begin{aligned} \omega &= \frac{1}{f(\alpha f^2 + \frac{z^2}{18})}\hat{K} \ , \qquad 3G^- + G^+ = -\frac{3}{f(\alpha f^2 + \frac{z^2}{18})}df \wedge \hat{K} \\ G^+ &= \frac{3}{2}d[\frac{1}{\alpha f^2 + \frac{z^2}{18}}\hat{K}] \ , \qquad G^- = -\frac{1}{2f^2}d[\frac{f^2}{\alpha f^2 + \frac{z^2}{18}}\hat{K}] \end{aligned} \tag{5.73}$$

To proceed we define the following vector fields;

$$\begin{aligned} S^i &= \frac{f}{2(\alpha f^2 + \frac{z^2}{18})}\nabla^i f \ , \quad (\sigma^1)^i = \frac{f}{(\alpha f^2 + \frac{z^2}{18})}(X^1)^{ij}\nabla_j f \\ (\sigma^2)^i &= (\alpha f^2 + \frac{z^2}{18})^{-\frac{1}{2}}(X^2)^{ij}\nabla_j f \ , \quad (\sigma^3)^i = (\alpha f^2 + \frac{z^2}{18})^{-\frac{1}{2}}(X^3)^{ij}\nabla_j f \end{aligned} \tag{5.74}$$

and we note that $\hat{K} = \frac{z}{6f}(\alpha f^2 + \frac{z^2}{18})\sigma^1$, so $\omega = \frac{z}{6f^2}\sigma^1$ and $G^- = -\frac{z}{12f^2}d(f\sigma^1)$. In addition, we note that the following constitutes an orthonormal basis of 1-forms;

$$e^1 = \sqrt{\frac{f}{2(\alpha f^2 + \frac{z^2}{18})}}df \ , \quad e^2 = \sqrt{\frac{(\alpha f^2 + \frac{z^2}{18})}{2f}}\sigma^1 \ , \quad e^3 = \sqrt{\frac{f}{2}}\sigma^2 \ , \quad e^4 = \sqrt{\frac{f}{2}}\sigma^3 \tag{5.75}$$

and so the metric on the base space is

$$ds^2 = \frac{f}{2(\alpha f^2 + \frac{z^2}{18})}df^2 + \frac{(\alpha f^2 + \frac{z^2}{18})}{2f}(\sigma^1)^2 + \frac{f}{2}((\sigma^2)^2 + (\sigma^3)^2) \tag{5.76}$$

where as a consequence of the integrability conditions the σ^i satisfy

$$d\sigma^1 = \sigma^2 \wedge \sigma^3 \ , \quad d\sigma^2 = \alpha\sigma^3 \wedge \sigma^1 \ , \quad d\sigma^3 = \alpha\sigma^1 \wedge \sigma^2 \ . \tag{5.77}$$

To continue, it is useful to introduce some local co-ordinates. In particular, we find from the integrability conditions that $[S, \hat{K}] = 0$ and so we can introduce local co-ordinates y and ϕ such that

$$S = \frac{\partial}{\partial y} \ , \qquad \hat{K} = \frac{\partial}{\partial \phi} \tag{5.78}$$

and let the remaining two co-ordinates be θ, ψ. In particular, as $S(f) = 1$ and $\hat{K}(f) = 0$, it follows that $f = y + Q(\theta, \psi)$. Moreover, we note that $i_S\sigma^i = i_{\hat{K}}\sigma^2 = i_{\hat{K}}\sigma^3 = 0$ and $i_{\hat{K}}\sigma^1 = \frac{z}{3}$ and therefore we find that $\mathcal{L}_S((\sigma^2)^2 + (\sigma^3)^2) = \mathcal{L}_{\hat{K}}((\sigma^2)^2 + (\sigma^3)^2) = 0$. Hence we can write

$$\begin{aligned} ((\sigma^2)^2 + (\sigma^3)^2) &= \mathcal{H}(\theta, \psi)^2(d\theta^2 + \sin^2\theta d\psi^2) \\ \sigma^1 &= \frac{z}{3}d\phi + \mathcal{P}(\theta, \psi)d\psi \end{aligned} \tag{5.79}$$

where $\mathcal{H}$, $\mathcal{P}$ are constrained by the integrability conditions. In particular, requiring that f^2G^- be covariantly constant together with the vanishing of the Ricci tensor implies

$$\frac{\partial \mathcal{P}}{\partial \theta} = \pm\mathcal{H}^2 \sin\theta \tag{5.80}$$

together with

$$\log \mathcal{H} = 1 - \alpha\mathcal{H}^2 \tag{5.81}$$

where denotes the Laplacian defined with respect to the 2-metric ${ds_2}^2 = d\theta^2 + \sin\theta^2 d\psi^2$. It is convenient to write the metric on the unit 2-sphere in terms of complex co-ordinates Z, $\bar{Z}$ where $Z = \cot\frac{\theta}{2}e^{i\psi}$, $\bar{Z} = \cot\frac{\theta}{2}e^{-i\psi}$;

$${ds_2}^2 = d\theta^2 + \sin\theta^2 d\psi^2 = \frac{4}{(1 + Z\bar{Z})^2}dZd\bar{Z} \tag{5.82}$$

then (5.81) can be written as $(1+Z\bar{Z})^2 \frac{\partial^2}{\partial Z \partial \bar{Z}} \log \mathcal{H} = 1 - \alpha \mathcal{H}^2$. On setting $\mathcal{H} = (1+Z\bar{Z})\mathcal{G}$ we observe that

$$\frac{\partial^2}{\partial Z \partial \bar{Z}} \log \mathcal{G} = -\alpha \mathcal{G}^2 \ . \tag{5.83}$$

There are therefore three cases to consider. In the first, $\alpha = 0$ and so $\mathcal{G} = e^{\mathcal{F}+\bar{\mathcal{F}}}$ where $\mathcal{F}(Z)$ is holomorphic in Z. Hence, by making a holomorphic co-ordinate transformation, we can set $\mathcal{G} = 1$ which corresponds to taking $\mathcal{H} = \sin^{-2} \frac{\theta}{2}$, $\mathcal{P} = \mp 2 \sin^{-2} \frac{\theta}{2}$. The base metric is

$$ds^2 = \frac{9}{z^2} f df^2 + \frac{z^2}{36f}(\frac{z}{3}d\phi \mp 2\sin^{-2}\frac{\theta}{2}d\psi)^2 + \frac{f}{2\sin^4\frac{\theta}{2}}(d\theta^2 + \sin^2\theta d\psi^2) \tag{5.84}$$

and it is straightforward to show that this metric is Gibbons-Hawking with tri-holomorphic Killing vector $\frac{\partial}{\partial \phi}$, which clearly preserves σ^1.

In the second case, $\alpha > 0$ and so the general solution to the Liouville equation (5.83) is $\mathcal{G}^2 = \alpha^{-1}(1+\mathcal{F}\bar{\mathcal{F}})^{-2}\frac{d\mathcal{F}}{dZ}\frac{d\bar{\mathcal{F}}}{d\bar{Z}}$ where $\mathcal{F}(Z)$ is holomorphic in Z. So by making a holomorphic change of co-ordinates we can set $\mathcal{H} = \frac{1}{\sqrt{\alpha}}$ which corresponds to $\mathcal{P} = \mp \alpha^{-1} cos\theta$. The metric on the base is then

$$ds^2 = \frac{f}{2}(\alpha f^2 + \frac{z^2}{18})^{-1} df^2 + \frac{f}{2\alpha}(d\theta^2 + \sin^2\theta d\psi^2) + \frac{1}{2f}(\alpha f^2 + \frac{z^2}{18})(\frac{z}{3}d\phi \mp \alpha^{-1} cos\theta d\psi)^2 \tag{5.85}$$

which is Gibbons-Hawking with tri-holomorphic Killing vector $\frac{\partial}{\partial \psi}$ which preserves σ^1.

In the last case, $\alpha < 0$, and so on setting $\beta = -\alpha$, the general solution to the Liouville equation (5.83) is $\mathcal{G}^2 = \beta^{-1}(\mathcal{F}+\bar{\mathcal{F}})^{-2}\frac{d\mathcal{F}}{dZ}\frac{d\bar{\mathcal{F}}}{d\bar{Z}}$ where $\mathcal{F}(Z)$ is holomorphic in Z. Hence, by making a holomorphic co-ordinate transformation, we can take $\mathcal{H} = \frac{1}{\sqrt{\beta}\sin\theta\cos\psi}$. Then the base metric is given by

$$ds^2 = \frac{f}{2}(-\beta f^2 + \frac{z^2}{18})^{-1} df^2 + \frac{f}{2\beta\sin^2\theta\cos^2\psi}(d\theta^2 + \sin^2\theta d\psi^2) + \frac{1}{2f}(-\beta f^2 + \frac{z^2}{18})(\frac{z}{3}d\phi + \mathcal{P}d\psi)^2 \tag{5.86}$$

where $\frac{\partial \mathcal{P}}{\partial \theta} = \pm \frac{1}{\beta \sin\theta\cos^2\psi}$. On making a change of co-ordinates $d\theta = \sin\theta d\chi$ together with a shift in ϕ this metric can be rewritten as

$$ds^2 = \frac{f}{2}(-\beta f^2 + \frac{z^2}{18})^{-1} df^2 + \frac{f}{2\beta\cos^2\psi}(d\chi^2 + d\psi^2) + \frac{1}{2f}(-\beta f^2 + \frac{z^2}{18})(\frac{z}{3}d\phi \mp \beta^{-1}\tan\psi d\chi)^2 \tag{5.87}$$

which is Gibbons-Hawking with tri-holomorphic Killing vector $\frac{\partial}{\partial \chi}$ which preserves σ^1. **Q.E.D.**

5.3.2 Maximally Supersymmetric Gibbons-Hawking Solutions

We have shown that in all cases the base space corresponding to the maximally supersymmetric timelike solutions is Gibbons-Hawking, and moreover, the tri-holomorphic Killing vector preserves f and ω. Hence, these solutions fall into the classification of Gibbons-Hawking solutions given in Section 3.7. It remains to examine the constraints imposed on the harmonic functions H, K, L and M by the integrability conditions.

To proceed, we note that (5.47) implies that

$$d(\frac{K}{H}) \wedge d(\frac{L}{H}) = 0 \tag{5.88}$$

We shall assume that $\frac{K}{H}$ is not constant, as if $\frac{K}{H}$ is constant then $G^+ = 0$ and the base is flat; we have already considered these solutions. Hence $\frac{L}{H} = \mathcal{F}(\frac{K}{H})$ for some function $\mathcal{F}$; in fact as a consequence of the harmonicity of L, H and K we have $L = \beta H + \gamma K$ for constants β, γ. In addition, it is clear that as K is defined only up to a shift of a multiple of H, we may without loss of generality set $\gamma = 0$, and so $L = \beta H$. To continue, we note that (5.48) implies that

$$d(\frac{K}{H}) \wedge d(M + \frac{\beta}{2}K) = 0 \tag{5.89}$$

and hence $M + \frac{\beta}{2}K = \mathcal{H}(\frac{K}{H})$ for some function $\mathcal{H}$ to be determined. However (5.48) also forces $\mathcal{H}$ to be constant, and so without loss of generality we obtain $M = -\frac{\beta}{2}K$. The remaining components of (5.48) together with (5.49) imply the following constraints of on H and K:

$$\begin{aligned} 2\rho\delta_{ij} &= \nabla_i \nabla_j [HK(\beta H^2 + K^2)^{-2}] \\ 2\chi\delta_{ij} &= \nabla_i \nabla_j [(K^2 - \beta H^2)(\beta H^2 + K^2)^{-2}] \ , \end{aligned} \tag{5.90}$$

where ∇ denotes the covariant derivative with respect to the flat metric on $\mathbb{R}^3$, and

$$\begin{aligned} \rho &\equiv -(\beta H^2 + K^2)^{-4}[2HK(\beta H^2 - K^2)(|\nabla K|^2 - \beta |\nabla H|^2) + (K^4 - 6\beta H^2 K^2 + \beta^2 H^4)\nabla H.\nabla K] \\ \chi &\equiv (\beta H^2 + K^2)^{-4}[(K^4 - 6\beta H^2 K^2 + \beta^2 H^4)(|\nabla K|^2 - \beta|\nabla H|^2) + 8\beta HK(K^2 - \beta H^2)\nabla H.\nabla K] \end{aligned} \tag{5.91}$$

Note that (5.90) imply that ρ and χ are constant, and it is straightforward to show that $\rho = 0$ iff $K = 0$. Given these constraints, all of the remaining integrability conditions then hold automatically.

So, setting $Y_1 = \rho r^2 + \lambda_i x^i + \sigma$ and $Y_2 = \chi r^2 + \mu_i x^i + \gamma$ for λ_i, σ, μ_i, γ constants, (5.90) imply

$$\begin{aligned} HK(\beta H^2 + K^2)^{-2} &= Y_1 \\ (K^2 - \beta H^2)(\beta H^2 + K^2)^{-2} &= Y_2 \end{aligned} \tag{5.92}$$

Hence (5.92) fixes H and K according to

$$K = \delta H \tag{5.93}$$

where δ satisfies

$$\delta^2 - \frac{Y_2}{Y_1}\delta - \beta = 0 \tag{5.94}$$

and H is given by

$$H^2 = \frac{\delta}{Y_1}(\beta+\delta^2)^{-2} = \frac{Y_1}{\delta(4\beta Y_1^2 + Y_2^2)} \ . \tag{5.95}$$

With these constraints the five-dimensional spacetime geometry is simplified. In particular,

$$\begin{aligned} f &= \frac{H}{\beta H^2 + K^2} \\ \omega_5 &= \frac{K}{H^2}(\beta H^2 + K^2) \\ \nabla \times \boldsymbol{\omega} &= 2\beta(K\nabla H - H\nabla K) \ . \end{aligned} \tag{5.96}$$

Using these identities, the five-dimensional metric can be written as

$$\begin{aligned} ds^2 &= -\beta[dx^5 - \beta^{-1}K(\beta H^2+K^2)^{-1}dt + (\chi_i - \beta^{-1}K(\beta H^2+K^2)^{-1}\omega_i)dx^i]^2 \\ &+ \beta^{-1}(\beta H^2+K^2)^{-1}(dt+\omega_i dx^i)^2 - (\beta H^2+K^2)d\mathbf{x}^2 \end{aligned} \tag{5.97}$$

for $\beta \neq 0$ and

$$ds^2 = H^2K^{-4}dt^2 + 2K^{-1}dt(dx^5 + \chi_i dx^i) - K^2 d\mathbf{x}^2 \tag{5.98}$$

for $\beta = 0$.

5.3.3 Classifying the Solutions

We shall neglect cases in which Y_2 or Y_1 vanish, or for which $Y_2 \propto Y_1$ as this corresponds to setting $G^+ = 0$, which we have already classified. To proceed we shall consider the cases $\beta = 0$ and $\beta \neq 0$ separately; in the following (r, θ, ϕ) are standard spherical polar co-ordinates on $\mathbb{R}^3$.

If $\beta = 0$ then from (5.92) it is clear that there are two possibilities. In the first,

$$K = m \ , \qquad H = n_i x^i \tag{5.99}$$

for m, n_i constants; $m \neq 0$ and n_i not all vanishing. By changing co-ordinates according as $x^i = m^{-1}\hat{x}^i$, $t = m^3\hat{t}$ and $x^5 = m^{-2}\hat{x}^5$ we can without loss of generality set $m = 1$; hence it is clear that this solution is the maximally supersymmetric plane wave.

Alternatively, one has

$$K = \frac{m}{r} \ , \qquad H = \frac{k}{r} + \frac{n_i x^i}{r^3} \tag{5.100}$$

for m, k, n_i constants, $m \neq 0$. The five dimensional metric can be written as

$$\begin{aligned} ds^2 &= m^{-4}r^4(\frac{k}{r} + \frac{n\cos\theta}{r^2})^2 dt^2 + 2m^{-1}rdt(dx^5 + (k\cos\theta - \frac{n\sin^2\theta}{r})d\phi) \\ &- m^2 r^{-2}(dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)) \ . \end{aligned} \tag{5.101}$$

It is then convenient to change co-ordinates as $\phi = \phi' - \frac{1}{m^3}t'$, $r = nr'$, $t = n^{-1}t'$. In these new co-ordinates the metric is

$$ds^2 = m^{-4}(1+k^2r'^2)dt'^2+2m^{-1}r'dt'(dx^5+k\cos\theta d\phi')-m^2r'^{-2}dr'^2-m^2(d\theta^2+\sin^2\theta d\phi'^2)\,. \quad (5.102)$$

If $k=0$ then this metric is $AdS_3 \times S^2$. If $k \neq 0$ then by a re-scaling of r' and x^5 we may without loss of generality set $k=1$, and the curvature invariants of this metric are unchanged from the case when $k=0$.

Next consider the cases when $\beta \neq 0$. Then H^2 and K^2 can be written as

$$\begin{aligned} H^2 &= -\frac{1}{2\beta}[\frac{Y_2}{4\beta Y_1^2+Y_2^2} \mp \frac{1}{\sqrt{4\beta Y_1^2+Y_2^2}}] \\ K^2 &= \frac{1}{2}[\frac{Y_2}{4\beta Y_1^2+Y_2^2} \pm \frac{1}{\sqrt{4\beta Y_1^2+Y_2^2}}]\,. \end{aligned} \quad (5.103)$$

It is useful to define $P_\pm = \sqrt{Y_2 \pm 2\sqrt{-\beta}Y_1}$. Then

$$\frac{(P_+ \pm P_-)^2}{P_+^2P_-^2} = 2[\frac{Y_2}{4\beta Y_1^2+Y_2^2} \pm \frac{1}{\sqrt{4\beta Y_1^2+Y_2^2}}] \quad (5.104)$$

Hence, if $\beta < 0$ then $P_\pm$ are real, and it follows that

$$\begin{aligned} H &= \frac{1}{2\sqrt{-\beta}}[\frac{1}{P_-} \mp \frac{1}{P_+}] \\ K &= \frac{1}{2}[\frac{1}{P_-} \pm \frac{1}{P_+}]\,, \end{aligned} \quad (5.105)$$

so $\frac{1}{P_\pm}$ must be harmonic. This then implies that there are two sub-cases. In the first

$$H = \frac{1}{\sqrt{-\beta}}[m+\frac{n}{r}]\,, \qquad K = [m-\frac{n}{r}]\,. \quad (5.106)$$

If there exists a point at which $H>0$ and $f>0$ then we require $\beta mn > 0$. So m and n have opposite sign, and the Taub-NUT base space has negative mass.

For this solution,

$$\chi_i dx^i = \frac{n}{\sqrt{-\beta}}\cos\theta d\phi\,, \qquad \omega_i dx^i = -4mn\sqrt{-\beta}\cos\theta d\phi \quad (5.107)$$

and hence the five-dimensional metric is

$$\begin{aligned} ds^2 &= -\beta[dx^5 + \frac{1}{4mn\beta}(mr-n)dt + \frac{m}{\sqrt{-\beta}}\cos\theta d\phi]^2 - \frac{r}{4mn\beta}[dt - 4mn\sqrt{-\beta}\cos\theta d\phi]^2 \\ &+ \frac{4mn}{r}[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)]\,. \end{aligned} \quad (5.108)$$

By changing co-ordinates as $x^5 = \frac{1}{\sqrt{-\beta}}(t' - n\psi)$, $t = 4mn\sqrt{-\beta}\psi$ and $r = -\frac{\rho^2}{16mn}$, (5.108) can be simplified as

$$ds^2 = [dt' + \frac{1}{4n}(d\psi - \cos\theta d\phi)]^2 - [d\rho^2 + \frac{\rho^2}{4}(d\theta^2 + \sin\theta^2 d\phi^2 + (d\psi - \cos\theta d\phi)^2)] \tag{5.109}$$

This is the Gödel solution.

In the second sub-case H and K have two poles given by $R_\pm = \sqrt{r^2 \pm 2\lambda r\cos\theta + \lambda^2}$

$$H = \frac{1}{\sqrt{-\beta}}[\frac{m}{R_+} + \frac{n}{R_-}] \ , \qquad K = \frac{m}{R_+} - \frac{n}{R_-} \tag{5.110}$$

for $\lambda > 0$ constant. Again, if there exists a point at which $H > 0$ and $f > 0$ then $\beta mn > 0$, so m and n have opposite sign. For this solution, it is most convenient to make the following changes of co-ordinate:

$$\begin{aligned} x^1 &= \lambda\sqrt{R^2-1}\sin\theta'\cos\phi' \ , \quad x^2 = \lambda\sqrt{R^2-1}\sin\theta'\sin\phi' \ , \quad x^3 = \lambda R\cos\theta' \\ x^5 &= \frac{1}{\sqrt{-\beta}}t' \ , \quad t = \frac{\sqrt{-\beta}}{\lambda}\psi \end{aligned} \tag{5.111}$$

and we obtain

$$\begin{aligned} H &= \frac{1}{\lambda\sqrt{-\beta}(R^2 - \cos^2\theta')}((m+n)R + (n-m)\cos\theta') \\ K &= \frac{1}{\lambda(R^2 - \cos^2\theta')}((m-n)R - (m+n)\cos\theta') \\ \chi_i dx^i &= \frac{1}{\sqrt{-\beta}(R^2 - \cos^2\theta')}((m+n)\cos\theta' R^2 + (m-n)\sin^2\theta' R - (m+n)\cos\theta')d\phi' \\ \omega_i dx^i &= -\frac{4mn\sqrt{-\beta}}{\lambda}\frac{(R^2-1)}{(R^2 - \cos^2\theta')}d\phi' \end{aligned} \tag{5.112}$$

and defining χ by $\chi = \phi' - \frac{\psi}{4mn}$ we obtain the metric

$$\begin{aligned} ds^2 &= (dt' + \frac{1}{4}(m^{-1} + n^{-1})\cos\theta' d\psi + (m-n)Rd\chi)^2 + 4mn(\frac{dR^2}{R^2-1} + (d\theta')^2) \\ &+ \frac{1}{4mn}\sin^2\theta'(d\psi)^2 + 4mn(R^2-1)d\chi^2 \ . \end{aligned} \tag{5.113}$$

Next consider the case when $\beta > 0$. Then $P_\pm$ are complex and

$$\begin{aligned} H &= \frac{1}{2\sqrt{-\beta}}[\frac{1}{P_-} - \frac{1}{P_+}] = \frac{1}{\sqrt{\beta}}\mathrm{Im}(\frac{1}{\mathrm{P}_-}) \\ K &= \frac{1}{2}[\frac{1}{P_-} + \frac{1}{P_+}] = \mathrm{Re}(\frac{1}{\mathrm{P}_-}) \ . \end{aligned} \tag{5.114}$$

Write $P_- = \sqrt{\tau r^2 + \Omega_i x^i + \nu}$ where τ, Ω_i and ν are generically complex constants. Requiring that $\frac{1}{P_-}$ be harmonic imposes the constraint $\tau\nu - \frac{1}{4}(\Omega_1^2 + \Omega_2^2 + \Omega_3^2) = 0$. There are again two sub-cases.

In the first, $\tau \neq 0$ and by making appropriate *real* shifts and rotations we can set

$$P_- = \zeta\sqrt{r^2 + 2i\lambda r\cos\theta - \lambda^2} \tag{5.115}$$

for $\zeta \in \mathbb{C}/\{0\}$ constant, and $\lambda > 0$ a real constant. Note that if $i\zeta \in \mathbb{R}$ then the harmonic function H corresponds to a singular Eguchi-Hanson base space. For this solution it is convenient to change coordinates as

$$\begin{aligned} x^1 &= \lambda\sqrt{R^2+1}\sin\theta'\cos\phi' \ , \quad x^2 = \lambda\sqrt{R^2+1}\sin\theta'\sin\phi' \ , \quad x^3 = \lambda R\cos\theta' \\ x^5 &= \frac{1}{\sqrt{\beta}}\psi \ , \quad t = \frac{\sqrt{\beta}}{\lambda}\alpha \end{aligned} \tag{5.116}$$

so that, on setting $\zeta = a + ib$ for $a, b \in \mathbb{R}$, $P_- = \zeta\lambda(R + i\cos\theta')$ and

$$\begin{aligned} H &= -\frac{1}{\sqrt{\beta}|\zeta|^2\lambda(R^2+\cos^2\theta')}(bR + a\cos\theta') \ , \quad K = \frac{1}{|\zeta|^2\lambda(R^2+\cos^2\theta')}(aR - b\cos\theta') \\ \chi_i dx^i &= \frac{1}{\sqrt{\beta}|\zeta|^2(R^2+\cos^2\theta')}(-b\cos\theta' R^2 + a\sin^2\theta' R - b\cos\theta')d\phi' \\ \omega_i dx^i &= \frac{\sqrt{\beta}}{\lambda|\zeta|^2}\frac{(R^2+1)}{(R^2+\cos^2\theta')}d\phi' \end{aligned} \tag{5.117}$$

and on defining t' by $t' = \phi' + |\zeta|^2\alpha$ the five dimensional geometry is given by

$$\begin{aligned} ds^2 &= -(d\psi + b\cos\theta' d\alpha - \frac{aR}{|\zeta|^2}dt')^2 + \frac{1}{|\zeta|^2}(R^2+1)(dt')^2 \\ &- \frac{1}{|\zeta|^2}(\frac{dR^2}{R^2+1} + (d\theta')^2) - |\zeta|^2\sin^2\theta'(d\alpha)^2 \ . \end{aligned} \tag{5.118}$$

In the second sub-case, $\tau = 0$ and without loss of generality we can set

$$P_- = \zeta\sqrt{r\sin\theta}e^{\frac{i\phi}{2}} \tag{5.119}$$

for $\zeta \in \mathbb{C}/\{0\}$ constant. By making a rotation, we can take $\zeta \in \mathbb{R}/\{0\}$. The metric for this case is given by taking

$$\begin{aligned} H &= -\frac{1}{\zeta\sqrt{\beta}}\frac{\sin\frac{\phi}{2}}{\sqrt{r\sin\theta}} \ , \qquad K = \frac{1}{\zeta\sqrt{r\sin\theta}}\cos\frac{\phi}{2} \ , \qquad \omega_i dx^i = -\zeta^{-2}\sqrt{\beta}r^{-1}\cot\theta dr \ , \\ \chi_i dx^i &= -\frac{1}{\zeta\sqrt{\beta}}r^{\frac{1}{2}}(\sin\theta)^{-\frac{3}{2}}(\cos\frac{\phi}{2}d\theta + \sin\theta\cos\theta\sin\frac{\phi}{2}d\phi) \ . \end{aligned} \tag{5.120}$$

In fact this solution is once more the maximally supersymmetric plane wave. To see this first note that by examining $HK(\beta H^2 + K^2)^{-2}$ it is clear that this solution has $\rho = 0$ and hence corresponds to one of the degenerate cases (5.84), (5.85) or (5.87) for which $K = 0$ as discussed previously. Moreover, the Ricci scalar of the five-dimensional geometry vanishes, and so the solution must correspond to (5.84), as this is the only case for which the Ricci scalar vanishes. Hence the five-dimensional geometry is given by

$$\begin{aligned} ds^2 &= f^2(dt + \frac{z}{6f^2}(\frac{z}{3}d\phi + 2\sin^{-2}\frac{\theta}{2}d\psi))^2 \\ &- f^{-1}[\frac{9}{z^2}f df^2 + \frac{z^2}{36f}(\frac{z}{3}d\phi + 2\sin^{-2}\frac{\theta}{2}d\psi)^2 + \frac{f}{2}\sin^{-4}\frac{\theta}{2}(d\theta^2 + \sin^2\theta d\psi^2)] \ . \quad (5.121) \end{aligned}$$

This metric is however equivalent to that given in (5.35) under the co-ordinate transformation $f = \frac{z}{3}x^1$, $x^2 = \sqrt{2}\cot\frac{\theta}{2}\cos\psi$, $x^3 = \sqrt{2}\cot\frac{\theta}{2}\sin\psi$, $t = \frac{3}{z}\hat{t}$ and $x^5 = \frac{z}{6}\phi - \psi$ with the identification $\boldsymbol{\omega} = (-\frac{3}{2}, 0, 0)$ with respect to Cartesian co-ordinates x^1, x^2, x^3. Hence the solution is the maximally supersymmetric plane wave.

5.4 Summary

In this section we have determined the most general solutions preserving maximal supersymmetry. We analysed the solutions that exist in the null class and the timelike class separately. In the null class we found flat space, the plane wave and $AdS_3 \times S^2$ and we subsequently saw that each of these also arise in the timelike class. The base space of the timelike class is always of Gibbons-Hawking (GH) type. Ignoring flat space, let us summarise our findings:

- Plane wave: this is in the null class and also in the time-like class, where it arises with a smeared Taub-NUT base space (see (5.99)). It also arose with the GH base space given in (5.120); it would be interesting to check whether or not this base space is distinct from smeared Taub-NUT.
- $AdS_3 \times S^2$: this is in the null class and also arises in the timelike class with a GH base with a dipole source (5.38).
- $AdS_2 \times S^3$: this has two timelike forms with base space given by flat space or singular negative Eguchi Hanson (see discussion following (5.57)).
- Generalised Gödel: this has two timelike forms with base space given by flat space or negative mass Taub-NUT (see the discussion following (5.64)).
- Near Horizon BMPV: this has a timelike form with flat base space.

In addition the timelike analysis revealed three more geometries with GH base spaces given in (5.102) with $k \neq 0$, (5.113) and (5.118). It seems plausible to us that these are all related to the BMPV solution. Strictly speaking we analysed necessary conditions for maximal supersymmetry and to confirm that these three geometries are indeed maximally supersymmetric solutions, one either needs to find a coordinate transformation confirming that they are indeed the BMPV solution, or perhaps another maximally supersymmetric solution listed above, all of which are known to be explicitly supersymmetric, or alternatively exhibit the Killing spinors directly.

6 G-Structures

We have obtained simple forms for all bosonic solutions of minimal $D = 5$ supergravity that preserve some supersymmetry. Following [12, 13], our method can be related to the notion of G-structures. Recall that a G-structure is a reduction of the principal frame bundle F with structure group $GL(n, \mathbb{R})$, to a subbundle P with structure group G (see eg [47]). Typically such a reduction is equivalent to the existence of certain globally defined tensors which are invariant under the group G and it is often convenient to refer to this set of tensors when talking about a G-structure. In the present setting we assume that the $D = 5$ manifold has a Lorentzian metric g and a spin structure and hence generically has a $Spin(1, 4)$ structure. The existence of a globally defined Killing spinor ϵ, with isotropy group $G \subset Spin(1, 4)$, gives rise to a G-structure. In particular, various G-invariant tensors can be formed from bilinears in the Killing spinor and these are equivalent to a G-structure.

There are two maximal subgroups of $Spin(1, 4)$ that leave a spinor invariant [46]. They are characterised by whether the corresponding vector built from the spinor is time-like or null. In the former case the subgroup is $SU(2)$ while in the latter case it is $\mathbb{R}^3$. In other words for the supersymmetric solutions admitting a Killing spinor giving rise to a time-like Killing vector the D=5 geometry admits an $SU(2)$ structure while if it gives rise to a null killing vector it gives rise to an $\mathbb{R}^3$ structure. In each case the G-structures are characterised by the algebraic properties satisfied by the metric g, the vector V and the two forms X^i, which we derived in section 2.

Actually, we should be a little more precise. In the null case, the vector is null everywhere and hence the $\mathbb{R}^3$ structure is indeed globally defined. However, in the timelike case the vector can become null, for example at the horizon of a black hole. Our analysis in section 2 was based on a neighbourhood where K was timelike. In this topologically trivial neighbourhood the Killing spinor defines an $SU(2)$ structure. This fact in itself is rather trivial since locally the frame bundle can always be trivialised. However, the Killing spinor defines a privileged $SU(2)$ structure satisfying certain differential conditions which are not trivial and in fact allow one to deduce the local form of the solution. The full solution can then be obtained by analytic continuation. Note that outside of regions where K becomes null it defines a global $SU(2)$ structure.

Following [46], these structures can be seen rather explicitly by exploiting the isomorphism $Spin(1, 4) \simeq Sp(1, 1)$. We realize $Spin(4, 1)$ as 2×2 quaternionic matrices A that satisfy $A^\dagger Q A = Q$, where

$$Q = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

The spinors are identified with vectors in $\mathbb{H}^2$ and the action on spinors is just given by matrix multiplication, $A \cdot s = As$. The two types of spinors together with the stabilizer groups are:

$$s = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{with stabilizer} \quad G = \left\{ \begin{pmatrix} 1+q & -q \\ q & 1-q \end{pmatrix} \Big| \; q \in Im\mathbb{H} \right\} \simeq \mathbb{R}^3 \tag{6.1}$$

and,

$$s = \begin{pmatrix} r \\ 0 \end{pmatrix} \quad \text{with stabilizer} \quad G = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & q \end{pmatrix} \Big| \; q \in Sp(1) \right\} \simeq SU(2) \; . \tag{6.2}$$

We can identify $\mathbb{R}^{4,1}$ with matrices of the form $m = \begin{pmatrix} t & q \\ \bar{q} & t \end{pmatrix}$, with $q = x_1 + ix_2 + jx_3 + kx_4 \in \mathbb{H}$. The norm of a vector is then given by $det(m) = t^2 - q\bar{q}$. Given a spinor $s = \begin{pmatrix} p \\ q \end{pmatrix}$ we can construct a vector $V(s)$ by:

$$V(s) = \begin{pmatrix} \frac{p\bar{p}+q\bar{q}}{2} & p\bar{q} \\ q\bar{p} & \frac{p\bar{p}+q\bar{q}}{2} \end{pmatrix} \; . \tag{6.3}$$

Using this explicit construction we see that the spinors with stabilizer $SU(2)$ give timelike vectors, while the ones with $\mathbb{R}^3$ give null vectors.

Lets consider first the timelike spinors which define an $SU(2)$-structure. Let $g_0 = dt^2 - \sum_{i=1}^4 dx_i^2$ be the standard Minkowski metric on $\mathbb{R}^{4,1}$ and let,

$$\begin{aligned} V_0 &= \partial_t \\ X_0^{(1)} &= dx^{12} - dx^{34} \\ X_0^{(2)} &= dx^{13} + dx^{24} \\ X_0^{(3)} &= dx^{14} - dx^{23} \; . \end{aligned} \tag{6.4}$$

The subgroup of $Spin(1,4)$ that leaves $(g_0, V_0, X_0^{(i)})$ invariant is $SU(2)_L \subset SU(2)_L \times SU(2)_R \simeq SO(4) \subset Spin(1,4)$. The three forms $X_0^{(i)}$ define an almost hyper-Kähler structure on the space transverse to the orbits of the vector V_0. A five dimensional manifold M is said to admit an $SU(2)$-structure if there exists a non-degenerate metric g, a vector V and three one forms $X^{(i)}$ such that at each point p there is a map $\alpha : T_pM \to \mathbb{R}^{4,1}$ under which $(g, V, X^{(i)})$ are identified with $(g_0, V_0, X_0^{(i)})$. Using these tensors one can consistently reduce the structure group to $SU(2)_L$.

Let us now discuss the null case. We saw above that we have an $\mathbb{R}^3$ structure in this case. Consider the metric $g_0 = 2dx^+dx^- - dx^i dx^i$ on $\mathbb{R}^{4,1}$ and let:

$$\begin{aligned} V_0 &= \partial_+ \\ X_0^{(i)} &= dx^- \wedge dx^i \; . \end{aligned} \tag{6.5}$$

A five dimensional manifold M is said to admit an $\mathbb{R}^3$ structure if there exist $(g, V, X^{(i)})$ such that at each point p there exists a map $\alpha : T_pM \to \mathbb{R}^{4,1}$ under which $(g, V, X^{(i)})$ are identified with $(g_0, V_0, X_0^{(i)})$. The action of $a^i \in \mathbb{R}^3$ on $T^*\mathbb{R}^{1,4}$ is given by:

$$\begin{aligned} dx^{-'} &= dx^- \\ dx^{+'} &= dx^+ + r^2 dx^- + \sqrt{2} a^i dx^i \\ dx^{i'} &= dx^i + \sqrt{2} a^i dx^- \end{aligned} \qquad (6.6)$$

where $r^2 = a^i a^i$. Given this explicit action it is clear that (g_0, V_0, X_0^i) are left invariant under the action of $\mathbb{R}^3$ and thus form an $\mathbb{R}^3$ structure.

The G-structures of interest here can be classified by taking the covariant derivative of the tensors defining the G-structure with respect to Levi-Civita connection and then decomposing into G-modules. Such a decomposition defines the intrinsic torsion of the G-structure. For example, if all of the G-modules vanish, which is equivalent to the tensors defining the G-structure being covariantly constant, then the Levi-Civita connection has holonomy contained in G. This is what occurs in the D=5 supersymmetric solutions for vanishing field strength: $\mathbb{R}^3$ and $SU(2)$ holonomy for the null and timelike cases, respectively. When the field strength is non-vanishing the $\mathbb{R}^3$ and $SU(2)$ structures are more general and their type is specified by the differential conditions imposed upon the tensors that we derived using the Killing spinor equation in section 2. For example, the vector V in both cases is not arbitrary but must be a Killing vector and, for the time-like case, the almost hyper-Kähler structure is actually integrable.

Since we were able to fully characterize the supersymmetric configurations by the conditions imposed on the tensors g, V, X^i we conclude that the types of G-structure that arise in each case provide both necessary and sufficient conditions for the existence of supersymmetric configurations of $D = 5$ minimal supergravity. This was also true for the class of solutions of $D = 10$ supergravity discussed in [12, 13].

7 The Gödel solution in $D = 11$ supergravity

All of the solutions of $N = 1$, $D = 5$ supergravity can be uplifted on a flat six space to obtain solutions of $D = 11$ supergravity. Perhaps the most surprising solution that we found is the maximally supersymmetric Gödel solution. Uplifting it to $D = 11$ gives another surprise: naively one would have expected it to still preserve 8 supersymmetries but in fact it preserves 20 supersymmetries.

The solution in $D = 11$ can be written

$$\begin{aligned} ds^2 &= -(dt + \omega)^2 + ds^2(\mathbb{E}^4) + ds^2(\mathbb{E}^6) \\ F &= -\gamma J \wedge K \end{aligned} \qquad (7.1)$$

where J, K are Kähler forms on $\mathbb{E}^4$, $\mathbb{E}^6$, respectively given by

$$\begin{aligned} J &= dx^1 \wedge dx^2 + dx^3 \wedge dx^4 \\ K &= dx^5 \wedge dx^6 + dx^7 \wedge dx^8 + dx^9 \wedge dx^\sharp \end{aligned} \tag{7.2}$$

and

$$\omega = \frac{\gamma}{2}(-x^2 dx^1 + x^1 dx^2 - x^4 dx^3 + x^3 dx^4) \tag{7.3}$$

and hence

$$d\omega \equiv \gamma J \ . \tag{7.4}$$

It is straightforward to show that this solves the equations of motion given by

$$\begin{aligned} R_{\mu\nu} - \frac{1}{12}(F_{\mu\sigma_1\sigma_2\sigma_3} F_\nu{}^{\sigma_1\sigma_2\sigma_3} - \frac{1}{12} g_{\rho\mu} F^2) &= 0 \\ d * F + \frac{1}{2} F \wedge F &= 0 \end{aligned} \tag{7.5}$$

where $\epsilon_{0123456789\sharp} = 1$. To determine the amount of supersymmetry we first note that the conventions we are using have $\Gamma_{0123456789\sharp} = 1$ and hence the Killing spinor equation is given by

$$\nabla_\mu \epsilon + \frac{1}{288}[\Gamma_\mu{}^{\nu_1\nu_2\nu_3\nu_4} - 8\delta_\mu^{\nu_1} \Gamma^{\nu_2\nu_3\nu_4}] F_{\nu_1\nu_2\nu_3\nu_4} \epsilon = 0 \ . \tag{7.6}$$

Next introduce the obvious orthonormal frame: $(dt + \omega), dx^i, dx^a$ with $i = 1, 2, 3, 4$, $a = 5, \ldots, \sharp$. It is useful to introduce spinors with three upper indices, $\epsilon^{\cdot\cdot\cdot}$, each taking the value $\pm$ which specify the chirality with respect to $\Gamma_{5678}, \Gamma_{789\sharp}$ and Γ_{056}:

$$\begin{aligned} \Gamma_{5678} \epsilon^{\pm\cdot\cdot} &= \pm \epsilon^{\pm\cdot\cdot} \\ \Gamma_{789\sharp} \epsilon^{\cdot\pm\cdot} &= \pm \epsilon^{\cdot\pm\cdot} \\ \Gamma_{056} \epsilon^{\cdot\cdot\pm} &= \pm \epsilon^{\cdot\cdot\pm} \ . \end{aligned} \tag{7.7}$$

We then find that the following constant chiral spinors are killing spinors:

$$\epsilon^{+++}, \qquad \epsilon^{+--}, \qquad \epsilon^{-++} \ . \tag{7.8}$$

In addition

$$\epsilon = \theta^{--+} + (1 - \gamma J_{ij} x^i \Gamma^{j56}) \theta^{---} \tag{7.9}$$

for constant θ^{--+}, θ^{---} are also Killing spinors. There are no other Killing spinors. Hence the solution admits precisely 20 Killing spinors corresponding to 5/8 supersymmetry.

The solution has topology $\mathbb{R}^{11}$ and has closed time-like curves. Note that one can dimensionally reduce this solution on the $x^\sharp$ direction to obtain a type IIA solution and then T-dualise to obtain a type IIB solution each of which preserves 20 supersymmetries. These solutions would involve Ramond-Ramond fields but their simplicity and high degree of symmetry suggests that it might be interesting to study string propagation in these backgrounds.

It is interesting to note that Gödel-like solutions of string theory have been obtained before. A class of exact string backgrounds with vanishing Ramond-Ramond fields was obtained in [48]. One of these solutions (equation 4.11) was interpreted as a rotating universe. String quantization in this background was studied in [49, 50]. Surprisingly, it was not noticed in any of these papers that the solution contains closed timelike curves or that it describes a Gödel-like solution.

8 Discussion

We have shown that any supersymmetric solution of minimal $N = 1$, $D = 5$ supergravity can be written in one of two simple forms. Solutions with Killing spinors giving rise to timelike vectors have $SU(2)$ structures while those giving rise to null Killing vectors have $\mathbb{R}^3$-structures. We have also determined the most general solutions preserving maximal supersymmetry. We have presented many new solutions but reasons of space have prevented us from analyzing most of them in any depth. It is obviously desirable to study our solutions more carefully to see if they contain any further surprises.

Our work generalizes the analysis of Tod [9] on the minimal $N = 2$, $D = 4$ theory, which can be obtained via dimensional reduction and truncation. The obvious next step is to undertake a similar analysis of the minimal $N = 1$, $D = 6$ supergravity theory, which also has 8 supercharges. In $D = 4$, all solutions could be obtained in explicit form but in $D = 5$ this is only possible in the null case, with the timelike case involving an arbitrary hyper-Kähler manifold. In $D = 6$ there is only a null case [46] and this seems to lead to the supersymmetric solutions exhibiting $SU(2) \ltimes \mathbb{R}^4$ structures. It will be interesting to see if integrable hyper-Kähler structures appear or whether something more general happens.

It would also be interesting to know the extent to which the method can be extended to non-minimal theories. Tod examined the minimal $N = 4$, $D = 4$ theory [10] (which is non-minimal in $N = 2$ language) but was unable to find all solutions, or provide (as we have) a simple algorithm for constructing solutions. However, since minimal $N = 1$, $D = 5$ reduces to minimal $N = 2$, $D = 4$ coupled to a vector multiplet, our results show that the latter theory must be tractable. Indeed, it is natural to conjecture that the dimensional reduction of our solutions with Gibbons-Hawking base space gives the entire timelike class of this $D = 4$ theory. The tractability of this theory suggests examining the general case of minimal $N = 2$, $D = 4$ coupled to arbitrarily many vector multiplets.

Gauged supergravities play an important role in the AdS/CFT correspondence, so a full understanding of the supersymmetric solutions of these theories is clearly desirable. As far as we know, no-one has examined this problem even for the simplest case of minimal $N = 2$, $D = 4$ gauged supergravity. We expect that this theory and the minimal $N = 1$, $D = 5$ gauged supergravity can both be analyzed using the techniques presented in this paper.

Our interest in understanding the general supersymmetric solutions of a higher dimensional supergravity theory was motivated in part by a desire to know whether there exist exotic supersymmetric black holes in five dimensions. Although we have not found any such solutions, our general solution is sufficiently complicated that it is not obvious that such solutions do not exist. It is clearly desirable to have a uniqueness theorem for supersymmetric black holes in order to justify the assumptions made in the black hole entropy calculations.[9] For the minimal $N = 1$, $D = 5$ theory, proving black hole uniqueness would involve showing that the only black hole solution belonging to our general solution is the BMPV solution. A first step might be to use asymptotic flatness to constrain the base space to be asymptotically Euclidean and therefore flat, if complete [51]. However, as we have seen, there is no reason to suppose that the base space has to be complete and once one permits incomplete metrics, there are many asymptotically Euclidean hyper-Kähler manifolds.

It is interesting that the maximally supersymmetric Gödel type solution of $D = 5$ supergravity lifts to a solution of $D = 11$ supergravity that preserves 20 supersymmetries. The simplicity of the solution suggests that there may well be other similar solutions preserving exotic fractions of supersymmetry. Of course the solution does have closed time like curves and thus its interpretation is not clear. More generally, one of the conclusions of the work presented here is that closed time-like curves are a commonplace amongst supersymmetric solutions. Perhaps there is a good reason why such solutions are not relevant in M-theory. On the other hand maybe they have a novel dual description waiting to be discovered.

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[9] We should note that such a theorem has not been proved even for the simplest theory admitting supersymmetric black holes, i.e., the minimal $N = 2$, $D = 4$ theory. This amounts to proving the conjecture [19] that the only black hole solutions in the IWP class are the Majumdar-Papapetrou multi-black hole solutions, which remains an open problem [52]. Of course, as soon as a four dimensional black hole is made arbitrarily non-extremal, the usual uniqueness theorems apply so this problem may not be of great physical interest, in contrast with the higher dimensional case.

A Conventions

We shall essentially use the conventions of [11], but it should be noted that unlike that reference our spinors are commuting spinors, throughout. The metric has signature $(+,-,-,-,-)$. Tangent space indices will be denoted $\alpha, \beta \ldots$ and curved indices by $\mu, \nu, \ldots$. The gamma matrices obey

$$\{\gamma_\alpha, \gamma_\beta\} = 2\eta_{\alpha\beta} \tag{A.1}$$

and satisfy

$$(\gamma_\alpha)^\dagger = \gamma^\alpha = \gamma_0 \gamma_\alpha \gamma_0. \tag{A.2}$$

The antisymmetrization of five gamma matrices is given by

$$\gamma_{\alpha\beta\gamma\delta\epsilon} = \epsilon_{\alpha\beta\gamma\delta\epsilon}, \tag{A.3}$$

where $\epsilon_{01234} = \epsilon^{01234} = +1$.

We use symplectic Majorana spinors ϵ^a_α, $a = 1, 2$, which are defined as follows. First let

$$\epsilon_a = \epsilon_{ab} \epsilon^b, \tag{A.4}$$

where ϵ_{ab} is antisymmetric with $\epsilon_{12} = 1$. It is convenient to introduce ϵ^{ab} such that $\epsilon^{12} = 1$. Now define

$$\bar{\epsilon}^a = {\epsilon_a}^\dagger \gamma_0. \tag{A.5}$$

The symplectic Majorana condition is

$$\bar{\epsilon}^a = \epsilon^{a\,t} C, \tag{A.6}$$

where the charge conjugation matrix is real and antisymmetric and satisfies

$$C \gamma^t_\alpha C^{-1} = \gamma_\alpha \tag{A.7}$$

Note that

$$\bar{\psi}^a \gamma_{\alpha_1 \ldots \alpha_m} \chi^b = -\bar{\chi}^b \gamma_{\alpha_m \ldots \alpha_1} \psi^a, \tag{A.8}$$

Given a spinor ϵ^a, one can construct bosonic quantities

$$X^{ab}_{\alpha_1 \ldots \alpha_p} \equiv \bar{\epsilon}^a \gamma_{\alpha_1 \ldots \alpha_p} \epsilon^b. \tag{A.9}$$

These quantities obey

$$X^{ab}_{\alpha_1 \ldots \alpha_p} = -X^{ba}_{\alpha_p \ldots \alpha_1} \tag{A.10}$$

as a consequence of equation (A.8). Furthermore,

$$\left(X^{ab}_{\alpha_1 \ldots \alpha_p}\right)^* = \epsilon_{ac} \epsilon_{bd} X^{cd}_{\alpha_1 \ldots \alpha_p}. \tag{A.11}$$

The Fierz identity is given by:

$$\bar{\epsilon}_1\epsilon_2\bar{\epsilon}_3\epsilon_4 = \frac{1}{4}\left(\bar{\epsilon}_1\epsilon_4\bar{\epsilon}_3\epsilon_2 + \bar{\epsilon}_1\gamma_\alpha\epsilon_4\bar{\epsilon}_3\gamma^\alpha\epsilon_2 - \frac{1}{2}\bar{\epsilon}_1\gamma_{\alpha\beta}\epsilon_4\bar{\epsilon}_3\gamma^{\alpha\beta}\epsilon_2\right). \quad \text{(A.12)}$$

Most of the algebraic identities we recorded in section 2 were obtained by using the Fierz identity with $\bar{\epsilon}_1 = \bar{\epsilon}^a$, $\epsilon_2 = \epsilon^d$, $\bar{\epsilon}_3 = \bar{\epsilon}^c$ and then setting in turn $\epsilon_4 = \epsilon^b$, $\gamma_\alpha\epsilon^b$ and $\gamma_{\alpha\beta}\epsilon^b$. The remaining identites were obtained using $\bar{\epsilon}_1 = \bar{\epsilon}^a$, $\epsilon_2 = \gamma_\alpha\epsilon^d$, $\bar{\epsilon}_3 = \bar{\epsilon}^c$ and $\epsilon_4 = \gamma_\beta\epsilon^b$.

In various places we parametrise the 3-sphere $SU(2)$ by Euler angles (θ, ϕ, ψ) with ranges $0 \leq \theta \leq \pi$, $0 \leq \phi \leq 2\pi$ and $0 \leq \psi < 4\pi$. The detailed parametrisation we use is described in more detail in Appendix A of [53]. The left-invariant or "right" one-forms on $SU(2)$ are given by

$$\begin{aligned}
\sigma_1^R &= -\sin\psi d\theta + \cos\psi\sin\theta d\phi \\
\sigma_2^R &= \cos\psi d\theta + \sin\psi\sin\theta d\phi \\
\sigma_3^R &= d\psi + \cos\theta d\phi
\end{aligned} \quad \text{(A.13)}$$

The right-invariant or "left" one-forms are given by

$$\begin{aligned}
\sigma_1^L &= \sin\phi d\theta - \cos\phi\sin\theta d\psi \\
\sigma_2^L &= \cos\phi d\theta + \sin\phi\sin\theta d\psi \\
\sigma_3^L &= d\phi + \cos\theta d\psi.
\end{aligned} \quad \text{(A.14)}$$

The superscript R (L) refers to the fact that the left (right) invariant one forms are dual to left (right) invariant vector fields ξ_i^R (ξ_i^L) which generate right (left) group actions. We will also refer to ξ_i^R as a right vector field and to ξ_i^L as a left vector field. The right vector fields are given by

$$\begin{aligned}
\xi_1^R &= -\cot\theta\cos\psi\partial_\psi - \sin\psi\partial_\theta + \frac{\cos\psi}{\sin\theta}\partial_\phi \\
\xi_2^R &= -\cot\theta\sin\psi\partial_\psi + \cos\psi\partial_\theta + \frac{\sin\psi}{\sin\theta}\partial_\phi \\
\xi_3^R &= \partial_\psi
\end{aligned} \quad \text{(A.15)}$$

and the left vector fields by

$$\begin{aligned}
\xi_1^L &= -\frac{\cos\phi}{\sin\theta}\partial_\psi + \sin\phi\partial_\theta + \cot\theta\cos\phi\partial_\phi \\
\xi_2^L &= \frac{\sin\phi}{\sin\theta}\partial_\psi + \cos\phi\partial_\theta - \cot\theta\sin\phi\partial_\phi \\
\xi_3^L &= \partial_\phi.
\end{aligned} \quad \text{(A.16)}$$

B Integrability condition

We record here an integrability condition obtained from the Killing spinor equation. Taking the second covariant derivative of the Killing spinor equation and anti-symmetrising we obtain:

$$\begin{aligned}\nabla_{[\rho}\nabla_{\mu]}\epsilon &= -\frac{1}{4\sqrt{3}}(\gamma_{[\mu}{}^{\nu_1\nu_2} + 4\gamma^{\nu_1}\delta^{\nu_2}_{[\mu})\nabla_{\rho]}F_{\nu_1\nu_2}\epsilon \\ &+ \frac{1}{48}(\gamma_{[\mu}{}^{\nu_1\nu_2} + 4\gamma^{\nu_1}\delta^{\nu_2}_{[\mu})(\gamma_{\rho]}{}^{\sigma_1\sigma_2} + 4\gamma^{\sigma_1}\delta^{\sigma_2}_{\rho]})F_{\nu_1\nu_2}F_{\sigma_1\sigma_2}\epsilon \end{aligned} \tag{B.1}$$

and hence

$$\begin{aligned}\frac{1}{8}R_{\rho\mu\nu_1\nu_2}\gamma^{\nu_1\nu_2}\epsilon &= -\frac{1}{4\sqrt{3}}(\gamma_{[\mu}{}^{\nu_1\nu_2} + 4\gamma^{\nu_1}\delta^{\nu_2}_{[\mu})\nabla_{\rho]}F_{\nu_1\nu_2}\epsilon \\ &+ \frac{1}{48}(-2F^2\gamma_{\mu\rho} - 4F^2_{\rho\nu}\gamma^\nu{}_\mu + 4F^2_{\mu\nu}\gamma^\nu{}_\rho + 12F_{\mu\nu_1}F_{\rho\nu_2}\gamma^{\nu_1\nu_2} \\ &+ 4F_{\nu_1\nu_2}F_{\nu_3\rho}\gamma_\mu{}^{\nu_1\nu_2\nu_3} - 4F_{\nu_1\nu_2}F_{\nu_3\mu}\gamma_\rho{}^{\nu_1\nu_2\nu_3})\epsilon \end{aligned} \tag{B.2}$$

where $F^2 \equiv F_{\mu\nu}F^{\mu\nu}$ and $F^2_{\mu\nu} \equiv F_{\mu\sigma}F_\nu{}^\sigma$. Now contracting both sides of this equation with γ^μ and using the Bianchi identity $R_{\mu[\nu\rho\sigma]} = 0$ we deduce the integrability condition:

$$\begin{aligned} 0 &= (R_{\rho\mu} + 2(F^2_{\rho\mu} - \frac{1}{6}g_{\rho\mu}F^2))\gamma^\mu\epsilon \\ &- \frac{1}{\sqrt{3}}\left[*(d*F + \frac{2}{\sqrt{3}}F\wedge F)\right]^\nu (2g_{\nu\rho} - \gamma_{\rho\nu})\epsilon \\ &- \frac{1}{6\sqrt{3}}dF_{\nu_1\nu_2\nu_3}(\gamma_\rho{}^{\nu_1\nu_2\nu_3} - 6\delta^{\nu_1}_\rho\gamma^{\nu_2\nu_3})\epsilon \end{aligned} \tag{B.3}$$

If we assume that a configuration admits Killing spinors and satisfies the equation of motion and the Bianchi identity for F we conclude that

$$E_{\mu\nu}\gamma^\nu\epsilon = 0 \tag{B.4}$$

where $E_{\mu\nu} = 0$ is equivalent to the Einstein equations. If we hit this with $\bar{\epsilon}$ we deduce that

$$E_{\mu\nu}V^\nu = 0 \tag{B.5}$$

On the other hand if we hit it with $E_{\mu\sigma}\gamma^\sigma$ we conclude that

$$E_{\mu\nu}E_\mu{}^\nu = 0 \qquad \text{no} \quad \text{sum} \quad \text{on} \quad \mu \tag{B.6}$$

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